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Hyperbolic Structures on Knot Complements

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1. INTRODUCTION

In the mid 1970s, 3-manifold topology was revolutionized by the ideas of Thurston. The main thrust of Thurston's ideas was that geometric structures existed on most 3-manifolds, and these geometries could be used to study the topology of 3-manifolds. By far the most complicated and interesting geometry is *hyperbolic geometry*, and the study of hyperbolic 3-manifolds has become a focal point for much recent work in 3-manifold topology. It is the intention of this paper to survey some of the applications of the existence of a hyperbolic structure on knot complements in S^3. The survey is for the most part self-contained, developing with the aid of examples, some of Thurston's ideas, and subsequent applications to knot theory.

The paper is organized as follows; in Section 2 we quickly review some of the basic theory of knots, their complements and fundamental groups. Section 2.3 consists of a collection of definitions and some terminology and it may be best to treat this section as reference whilst reading the paper. In Section 3 we review some hyperbolic geometry, and in Section 4 we give some discussion on the computation of volume in hyperbolic 3-space. Section 5 discusses hyperbolic structures on knot complements, culminating in a statement of Thurston's remarkable theorem (Theorem 5.1). The remaining sections, Sections 6–10, concern applications of the existence of a hyperbolic structure. We make no pretensions towards completeness, but rather the paper merely intends to survey some of the many applications of hyperbolic geometry in 3-manifolds.

2. KNOTS AND THEIR COMPLEMENTS

In this section we bring together some of the basic facts from knot theory we shall make use of, we refer the reader to [7], [14] or [57] for further details.

2.1

By a knot in S^3 we shall mean a smooth embedding of a circle in S^3, with $S^3\backslash K$ denoting the *knot complement* (it is often useful to work in $\mathbb{R}^3$ and view S^3 as the one point compactification of $\mathbb{R}^3$). It will sometimes be convenient to work with the *knot exterior*, $X_K = S^3\backslash\text{Int}(\eta(K))$, where $\eta(K)$ is a tubular neighborhood of K. The only difference with the knot complement is that X_K is a compact 3-manifold with boundary consisting of a torus. The fundamental groups of the knot complement and knot exterior are isomorphic groups (which are also isomorphic to $\pi_1(\mathbb{R}^3\backslash K)$), which we simply call the knot group of K, and denote it by $\pi_1(S^3\backslash K)$. $\eta(K)$ is a solid torus, and by a *meridian* of K we mean any simple closed curve on

$\partial\eta(K)$ which is nullhomologous in $\eta(K)$ but not in $\partial\eta(K)$. Any two meridians (suitably oriented) are isotopic. By a *longitude* for K we mean any simple closed curve on $\partial\eta(K)$ which intersects some meridian transversely in a single point. It is usual to have a preferred longitude, namely the unique longitude that is nullhomologous in X_K. If $\mathcal{M}$ is a meridian of K and ℓ a longitude meeting $\mathcal{M}$ as described, then the pair $\{\mathcal{M},\ell\}$ determines a *framing* of ∂X_K and any essential (does not bound a disc in ∂X_K) simple closed curve on ∂X_K is described as $\mathcal{M}^p\ell^q$ for coprime integers p and q. The subgroup $\langle\mathcal{M},\ell\rangle$ of $\pi_1(S^3\backslash K)$, is called a *peripheral subgroup*.

We will occasionally require the notion of a *link* in S^3, and by this we simply mean a finite collection of disjoint smoothly embedded circles in S^3. Much of what is discussed for knots has obvious reformulations for links.

2.2

We briefly recall the construction of the *Wirtinger presentation* of $\pi_1(S^3\backslash K)$. For this we view $K \subset \mathbb{R}^3$. Begin by projecting K onto the x–y plane P, yielding a finite collection of arcs $\alpha_1,\ldots, \alpha_n$ as shown in Fig. 1.

The projection should be regular in the sense that there are only a finite number of multiple points, and these multiple points are double points. Each α_i is assumed connected to α_{i-1} and α_{i+1} mod n, as shown in Fig. 1. We equip K with an orientation in such a way that the arcs α_i become oriented arcs, with the orientation compatible with the order of the subscripts.

Now fix a basepoint * above the projection plane P, say the point (0,0,1) and form n oriented loops $x_1,\ldots, x_n$, each based at * which pass under $\alpha_1,\ldots, \alpha_n$ respectively. Figure 1 shows the loops x_i represented by a short arrow under the arc α_i. The loop is completed in the obvious way by viewing the loop as the oriented triangle from * to the tail of x_i, traversing x_i and returning to *.

At each crossing in the projection of K, there is a relation among the x_i's which has the form given by one of the possibilities shown in Fig. 2.

With this notation we have the following theorem describing the Wirtinger presentation of the knot group.

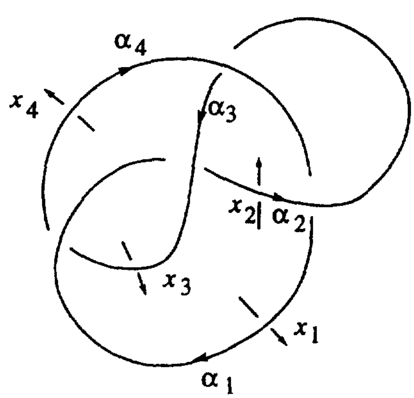
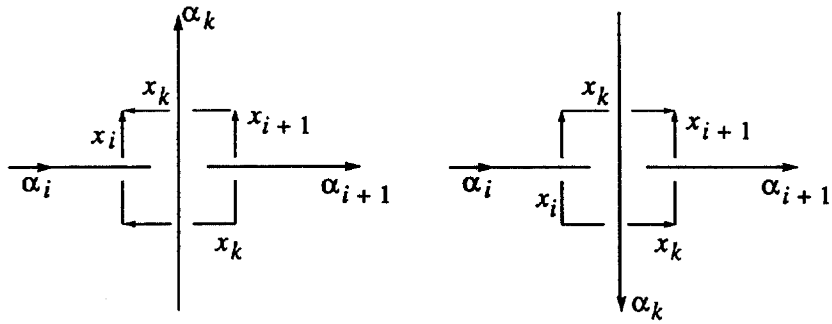


Fig. 1



r_i $\quad x_k x_i = x_{i+1} x_k \quad\quad x_i x_k = x_k x_{i+1}$

Fig. 2

Theorem 2.1. *$\pi_1(S^3 \backslash K)$ is generated by the (homotopy classes of the) x_i and has a presentation:*

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle,$$

where r_i is as in Fig. 2. Moreover, any one of the r_i may be omitted and the above remains true. □

Example. Referring to Fig. 1, the discussion above yields the following as relations for the figure-eight knot group:

$$x_1 x_3 = x_3 x_2, \tag{1}$$

$$x_4 x_2 = x_3 x_4, \tag{2}$$

$$x_3 x_1 = x_1 x_4. \tag{3}$$

We can use (1) and (3) to eliminate $x_2 = x_3^{-1} x_1 x_3$ and $x_4 = x_1^{-1} x_3 x_1$. Substituting in (2) gives the equivalent presentation:

$$\langle x_1, x_3 \mid x_1^{-1} x_3 x_1 x_3^{-1} x_1 x_3 = x_3 x_1^{-1} x_3 x_1 \rangle.$$

As a final remark in this subsection we note that all meridians are conjugate in $\pi_1(S^3 \backslash K)$.

2.3

We conclude this section by recalling some terminology. Two knots K and K' are *equivalent* if there exists a homeomorphism h: $S^3 \to S^3$ such that $h(K) = K'$. A knot K is non-trivial if it is not equivalent to a standardly embedded circle (the unknot) in S^3. If two knots are equivalent, this implies that their complements are homeomorphic. The following result of Gordon and Luecke [21] provided the converse of this in the context of prime knots. A knot K is called *composite* or the *connect sum* of non-trivial knots K_1 and K_2 if there is a 2-sphere S in S^3 meeting K in two points, dividing K into K_1 and K_2, see Fig. 3(a). Otherwise a knot is called *prime.*

Theorem 2.2. If K and K' are prime knots in S^3 which have homeomorphic complements then they are equivalent. □

Remark. Thus the fundamental group completely determines the knot. Unfortunately, the fundamental group is a rather intractable object to use as an invariant of the knot. We shall return to this point in the guise of hyperbolic structures.

A knot is called *alternating* if it admits a projection in which the crossings alternate over–under upon traversing the knot in a fixed direction. A *torus knot* is one that can be embedded as a simple closed curve on a standardly embedded (i.e. unknotted) torus in S^3.

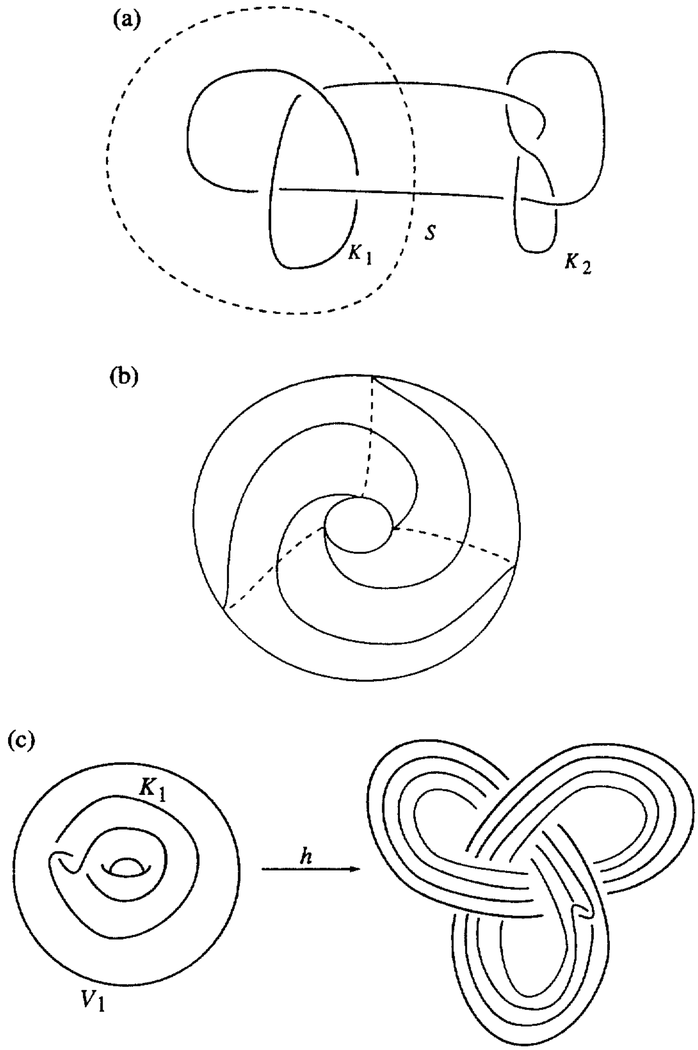


Fig. 3

More precisely, the torus knot $T_{p,q}$ of type p,q is the knot which wraps around the standard solid torus p times in the longitudinal direction, and q times in the meridinal direction. Figure 3(b) shows the trefoil knot, which is the torus knot $T_{2,3}$.

To define a *satellite knot* we proceed as follows; refer to Fig. 3(c). Let V_1 be an unknotted solid torus in S^3, and $K_1 \subset V_1$ a knot. Now knot the solid torus V_1 as shown; more precisely let h be a homeomorphism $V_1 \to V$ onto a tubular neighborhood V of a non-trivial knot K_2. The knot K obtained as $h(K_1)$ is called a satellite knot, and K_2 its companion.

If $S \neq S^2$ is an orientable surface properly embedded in X_K (so that $\partial S \subset \partial X_K$), S is called *incompressible*, if the induced map $\pi_1(S \hookrightarrow \pi_1(S^3 \backslash K)$ is injective. Otherwise the surface is called *compressible.*

If K is a non-trivial knot the torus ∂X_K is embedded and *incompressible.* For the unknot, the exterior is simply a solid torus in which case the boundary torus is compressible. Notice that in the exterior of a satellite knot K there is an embedded torus T_K which is the boundary of the knotted solid torus V_1. The torus T_K is also incompressible, but is 'distinct' from ∂X_K in

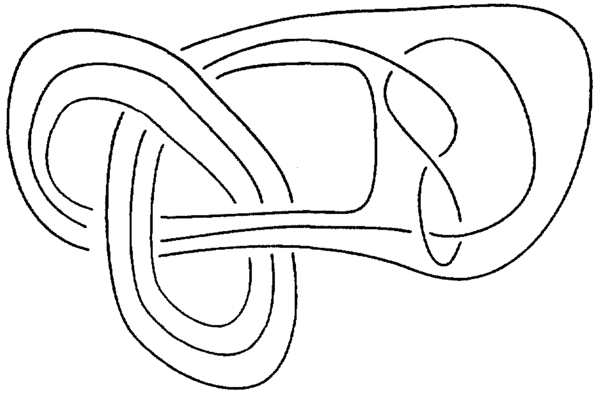


Fig. 4

the sense that by construction T_K is not parallel to ∂X_K. Algebraically this means $\pi_1(T_K)$ is not conjugate in $\pi_1(S^3\backslash K)$ to $\pi_1(\partial X_K)$.

If K is a composite knot the exterior admits an embedded torus, the so-called 'swallow-follow' torus as it swallows one of the factor knots and follows the other, see Fig. 4. The swallow-follow torus is always incompressible if the knots in the composition are non-trivial. Thus composite knots are subsumed in the family of satellite knots.

The *crossing number* of a knot K is the minimal number of crossings in a projection of the knot in a projection plane. A knot K admits an *m-bridge presentation* if there exists a 2-sphere S embedded in S^3 meeting K transversely in $2m$ points and dividing S^3 into two 3-balls B_i, such that $K \cap B_i$ consists of arcs which are unknotted, unlinked and trivially embedded in B_i. The *bridge number* of K is the minimal integer n for which K admits an n-bridge presentation.

By a *Seifert surface* for a knot K we mean a connected, orientable surface in S^3 with $\partial S = K$. The *genus* of a knot, is the minimal genus of an orientable Seifert surface for the knot.

3. HYPERBOLIC STRUCTURES

Here we shall summarize some of the material from hyperbolic geometry, and discrete groups that we shall need. We refer the reader to [12], [48] or [62] for details.

3.1

Hyperbolic n-space, $\mathbb{H}^n$ is the upper half space $\{x \in \mathbb{R}^n : x_n > 0\}$ in $\mathbb{R}^n$ equipped with the metric defined by the length element.

$$ds^2 = \frac{dx_1^2 + \dots + dx_n^2}{x_n^2}.$$

With this metric, $\mathbb{H}^n$ is a complete Riemannian manifold all of whose sectional curvatures are -1. Moreover $\mathbb{H}^n$ is the unique connected, simply connected, complete Riemannian manifold of constant curvature -1. Geodesics in this metric space are straight-lines and semi-circles orthogonal to the sphere-at-infinity $\{x \in \mathbb{R}^n : x_n = 0\} \cup \infty$, denoted in what follows by S_∞^{n-1}.

The group of isometries is denoted $\text{Isom}(\mathbb{H}^n)$ and is a real Lie group. In low dimensions there are isomorphisms $\text{Isom}_+(\mathbb{H}^2) \cong \text{PSL}(2,\mathbb{R})$ and $\text{Isom}_+(\mathbb{H}^3) \cong \text{PSL}(2,\mathbb{C})$. It is this latter group that is of most interest to us.

A *hyperbolic structure* on an n-manifold M is a Riemannian metric on M such that every point in M has a neighborhood isometric to an open subset of hyperbolic n-space. We will also have cause to speak about *incomplete* hyperbolic metrics on 3-manifolds, we will refer to this as incomplete hyperbolic structures. Hopefully there will be no confusion.

If Γ is a discrete, torsion-free subgroup of $\text{Isom}_+(\mathbb{H}^n)$, then Γ acts discontinuously and freely on $\mathbb{H}^n$ and so $\mathbb{H}^n/\Gamma$ admits a hyperbolic structure. More generally given any hyperbolic structure on an orientable manifold M there is a homomorphism of $\pi_1(M)$ into $\text{Isom}_+(\mathbb{H}^n)$ called the *holonomy* representation associated to the hyperbolic structure. This homomorphism is defined up to conjugacy as follows.

Any hyperbolic structure on M induces one on the universal cover $\tilde{M}$. There is a local isometry, called the *developing map*,

$$D:\tilde{M} \to \mathbb{H}^n,$$

defined by choosing a small open set, U, in $\tilde{M}$ and identifying it via an isometry, ϕ_U, with an open set in $\mathbb{H}^n$. One then extends this map by analytic continuation to all of $\tilde{M}$. This depends on an arbitrary choice of ϕ_U and it is easy to see that a different choice results in a developing map which differs from the first choice by composition with an isometry of $\mathbb{H}^n$. The developing map provides a homomorphism

$$\rho: \pi_1(M) \to \text{Isom}_+(\mathbb{H}^n)$$

which is the holonomy representation of the hyperbolic structure. The group $\rho(\pi_1(M))$ coincides with Γ in the special case described above. Covering transformations of $\tilde{M}$ are conjugate, via D to isometries of $\mathbb{H}^n$.

The restriction that a group act freely can be removed without too much loss. A quotient of $\mathbb{H}^n$ which arises in this way is called a *hyperbolic n-orbifold.*

It is a deep theorem of Mostow and Prasad ([40] and [47]) that if a compact orientable n-manifold $n \geq 3$, admits a hyperbolic structure of finite volume then this structure is unique. An equivalent reformulation says that if Γ_1 and Γ_2 are discrete isomorphic subgroups of $\text{Isom}_+(\mathbb{H}^n)$ with $\mathbb{H}^n/\Gamma_1$ and $\mathbb{H}^n/\Gamma_2$ finite volume, then Γ_1 and Γ_2 are conjugate in $\text{Isom}(\mathbb{H}^n)$. A corollary of this is the following: we denote the hyperbolic volume of M by $\text{Vol}(M)$.

Corollary 3.1. Let M a compact orientable n-manifold, $n \geq 3$ admitting a complete hyperbolic structure of finite volume. Then Vol(M) is a topological invariant of M. □

3.2

We will now expand on some of the above discussion in dimension 3. By a hyperbolic 3-manifold we will now always mean an orientable 3-manifold equipped with a complete Riemannian metric of constant curvature -1. A discrete subgroup of $\text{PSL}(2,\mathbb{C})$ is called a *Kleinian group.* Thus a hyperbolic 3-manifold M is identified with $\mathbb{H}^3/\Gamma$ where Γ is a torsion-free Kleinian group. Since the faithful discrete representation of $\pi_1(M)$ into $\text{PSL}(2,\mathbb{C})$ can be lifted to a faithful discrete representation into $\text{SL}(2,\mathbb{C})$ (cf. [62] chapter 5) it is often convenient to view elements of Kleinian groups as matrices. An elementary but important fact from the structure of subgroups of $\text{SL}(2,\mathbb{C})$ is the following lemma (see [48])

Lemma 3.2. Let Γ be a discrete subgroup of $SL(2,\mathbb{C})$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Then Γ is conjugate in $SL(2,\mathbb{C})$ to a subgroup of $\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{C}\right\}$. □

An element conjugate in SL(2,ℂ) to one of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is called *parabolic*. If Γ is a torsion-free Kleinian group then it is known that any $\mathbb{Z} \oplus \mathbb{Z}$ subgroup is *peripheral*, that is conjugate in Γ to $\pi_1(T)$ where T is a boundary component of $\mathbb{H}^3/\Gamma$. If $M = \mathbb{H}^3/\Gamma$ is a closed hyperbolic 3-manifold, there is a lower bound to the length of the shortest closed geodesic in M, and this is easily seen to imply (from the definition of the hyperbolic metric) that Γ cannot contain parabolic elements. Summarizing this discussion, we have:

Theorem 3.3. Let $M = \mathbb{H}^3/\Gamma$ be a finite volume hyperbolic 3-manifold. Then

- *if M is closed, Γ contains no $\mathbb{Z} \oplus \mathbb{Z}$ subgroup or,*
- *M is the interior of a compact manifold with boundary which consists of a disjoint union of a finite number of tori. Furthermore, any $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of Γ is peripheral.* □

Additionally, if M is as above and $\gamma \in \Gamma$ is not parabolic then, γ is called *loxodromic*. In terms of the action on $\mathbb{H}^3$, γ preserves a unique geodesic, A_γ in $\mathbb{H}^3$, called the *axis* of γ. γ acts by translating along A_γ by a distance $l(\gamma)$, and rotating by some angle θ, $0 \le \theta \le \pi$ about A_γ. On projecting to M, we obtain a closed geodesic of length $l(\gamma)$ in M. We remark that the elements which commute with γ are precisely those with axis A_γ. The following fact about Kleinian groups follows from this discussion.

Lemma 3.4. The center of a Kleinian group of finite covolume is trivial. □

We finish this section by quoting a standard result from Kleinian groups. Let Γ be a Kleinian group, and

$$\mathrm{Norm}(\Gamma) = \{g \in \mathrm{Isom}(\mathrm{H}^3) \mid g\Gamma g^{-1} = \Gamma\}$$

the normalizer of Γ in $\mathrm{Isom}(\mathbb{H}^3)$.

Theorem 3.5. Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold of finite volume. Then Norm(Γ) is a Kleinian group and $\mathbb{H}^3$/Norm(Γ) has finite volume. □

4. COMPUTING HYPERBOLIC VOLUMES

We describe here some features on the computation of volumes of hyperbolic 3-manifolds (see [35] and [62], chapter 7, for more details.

4.1

The Lobachevsky function $Л(\theta)$ is defined by the formula:

$$Л(\theta) = -\int_0^\theta \log |2 \sin u| du.$$

For practical calculations, the following series which converges for $|\theta| \le \pi$ is useful,

$$Л(\theta) = \theta\left(1 - \log |2\theta| + \sum \frac{B_n(2\theta)^{2n}}{2n(2n+1)!}\right),$$

where B_n is the nth Bernoulli number.

4.2

To relate the Lobachevsky function to the calculation of hyperbolic volume, we discuss *ideal tetrahedra* in hyperbolic 3-space.

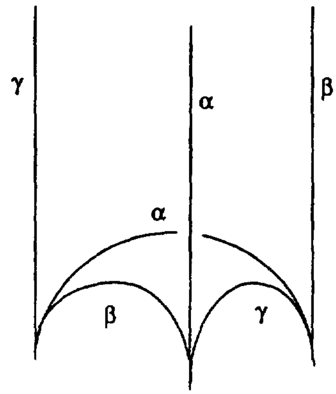


Fig. 5

Definition 4.1. By an ideal polyhedron we mean a polyhedron in $\mathbb{H}^3 \cup S^2_\infty$ all of whose vertices lie on the sphere-at-infinity, and all of whose edges are hyperbolic geodesics.

Figure 5 shows an ideal tetrahedron which will be of most interest to us.

A *horosphere* $\mathcal{H}$ in $\mathbb{H}^3$, is defined to be the intersection in $\mathbb{H}^3$ of a Euclidean sphere in $\mathbb{H}^3 \cup S^2_\infty$ tangent to S^2_∞ at $p \in S^2_\infty$. p is referred to (confusingly perhaps) as the *center* of $\mathcal{H}$. The interior of a horosphere is a *horoball*. When p is the point at ∞, a horosphere is just a horizontal plane at some height t up the x_3-axis.

If T is an ideal tetrahedron, a horospherical cross-section of T obtained by truncating T by a horosphere centered at the ideal vertex v cuts out a Euclidean triangle $L(v)$, most conveniently seen by locating v at ∞ via an element of PSL(2,$\mathbb{C}$). As Thurston describes in chapter 4 of [62], $L(v)$ determines T up to isometry, and moreover, T is completely determined up to isometry by the dihedral angles α, β and γ along edges incident to v with $\alpha+\beta+\gamma=\pi$. It also follows from relations between angles at the other vertices that the dihedral angles on opposite edges of T are equal (see [62], chapter 4). Thus T has the form shown in Fig. 5.

To compute the volume of T we have [62], chapter 7 or [35].

Theorem 4.2. The hyperbolic volume of an ideal tetrahedron with dihedral angles α, β and γ with $\alpha+\beta+\gamma=\pi$ is

$$Л(\alpha)+Л(\beta)+Л(\gamma). \qquad \square$$

Of particular interest is the ideal tetrahedron all of whose dihedral angles are $\pi/3$. In what follows this will be denoted $\mathcal{T}_0$. One of the reasons for its importance in the theory of hyperbolic 3-manifolds is (see [35]),

Theorem 4.3. The maximal possible volume of an ideal hyperbolic tetrahedron occurs for $\mathcal{T}_0$ with volume $v_0 = 3Л(\pi/3)$, which is approximately 1.0149416.... □

5. HYPERBOLIC STRUCTURES AND 3-MANIFOLD TOPOLOGY

The seminal work of Thurston (see [62, 63]) involved the deep insight that geometry had a fundamental role to play in understanding the topology of 3-manifolds. Recall that in dimension 2 a classical theorem states that every closed orientable 2-manifold is 'uniformizable', that is if S is a closed orientable 2-manifold, then S is homeomorphic to X/Γ where X is the 2-sphere S^2, the Euclidean plane $\mathbb{E}^2$ or the hyperbolic plane $\mathbb{H}^2$ and Γ a discrete group of isometries of X. Thurstons work led to a program in dimension 3 that is analogous to this, the so-called *Uniformization or Geometrization Conjecture*. We will not go into this here, but will content ourselves by saying that 'most' compact orientable 3-manifolds are conjectured to admit a hyperbolic structure and will only expand on this in the context of knot complements. However we begin with a discussion and some examples, in particular Example 2, due to Riley [53], that has been fundamental in the development of the subject.

5.1 *Example 1: The unknot*

If K denotes the unknot, then $\pi_1(S^3 \backslash K)$ is isomorphic to $\mathbb{Z}$ and so we can equip the knot complement with a hyperbolic structure by simply identifying this infinite cyclic group with a subgroup $\langle\gamma\rangle$ of PSL(2,**C**) generated by a unique loxodromic element γ. However, regardless of how one does this, the resulting hyperbolic structure is always of *infinite* volume.

If a knot complement admits a hyperbolic structure, then we deduce from Lemma 3.2 that a peripheral subgroup consists of parabolic elements, and hence as all meridians are conjugate, every meridian maps to a parabolic element under the holonomy representation into SL(2,$\mathbb{C}$). This is an example of what Riley called a *parabolic representation* or *p-rep* for short. Completely independently of hyperbolic structures, in the late 1960s and 1970s Riley (see [52], [53], and [54]) was searching for p-reps of knot groups into PSL(2,$\mathbb{C}$), by simply starting from a Wirtinger presentation and looking for homomorphisms of the knot group into PSL(2,$\mathbb{C}$) where the meridians map to parabolic elements. This work led (inadvertently) to the following important example [53] (one of many Riley computed).

5.2 *Example 2: The figure-eight knot*

From Section 2.2 we have the following presentation for the figure-eight knot group obtained by simplifying the Wirtinger presentation.

$$\langle x_1, x_3 \mid x_1^{-1}x_3x_1x_3^{-1}x_1x_3 = x_3x_1^{-1}x_3x_1\rangle.$$

To exhibit a hyperbolic structure on the complement of the figure-eight knot, Riley produces [53] a faithful discrete representation of Γ (as above) into PSL(2,$\mathbb{C}$). Let $\omega = \dfrac{(-1+\sqrt{-3})}{2}$, and define a map ϕ from Γ into SL(2,$\mathbb{C}$) by

$$\phi(x_1) = A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \phi(x_3) = B = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}.$$

That this defines a homomorphism can be checked by showing the relation in Γ is satisfied by the matrices A and B. To determine faithfulness and discreteness Riley constructs a fundamental polyhedron for the action of $\phi(\Gamma)$ on $\mathbb{H}^3$. Standard results in Kleinian groups imply the identification space of this polyhedron obtained by certain face pairings provide a hyperbolic 3-manifold of finite volume. To show that this manifold is homeomorphic to the complement of the figure-eight knot, Riley invokes the work of Waldhausen [65], by observing that the quotient manifold has the same fundamental group and peripheral structure as the figure-eight knot complement.

5.3 *Example 3: Torus and satellite knots*

It is not too hard to see given our discussion Sections 2.3 and 3.2 that the complement of a torus knot or satellite knot cannot admit a hyperbolic structure. For a torus knot it is known that the complement admits a so-called Seifert fibration and so in particular the knot group has a non-trivial center (see [14]) which is impossible for a Kleinian group by Lemma 3.4. For a satellite knot K, as discussed in Section 2.3 the knot group $\pi_1(S^3\backslash K)$ contains a $\mathbb{Z}\oplus\mathbb{Z}$ subgroup that is not peripheral. However, this is excluded by the existence of a hyperbolic structure by Theorem 3.3.

As discussed above, Riley's work on p-reps led him to discover the existence of hyperbolic structures on many knot complements. He was also aware of the obstructions to a hyperbolic structure on the complement of a torus or satellite knot as discussed in Example 3 above. From this he was able to guess that apart from these obstructions, the complements of prime knots should admit a hyperbolic structure of finite volume. However it was the remarkable pioneering work of Thurston that provided the deep machinery to establish the following theorem which shows that the obstructions above are the only obstruction to a hyperbolic structure, see [63], [37] for a discussion of the proof. A complete proof of Thurston's hyperbolization theorem, of which Theorem 5.1 is a special case, has been written up by Otal [44, 45]. An amusing personal account by Riley is given in [56] where he describes the development of his work till the time he met Thurston.

Theorem 5.1. Let $K \subset S^3$ be a prime non-trivial knot. Then $S^3\backslash K$ has a hyperbolic structure of finite volume if and only if K is not a torus knot or a satellite knot. □

The Six Smallest Known Knot Complements in the 3-Sphere

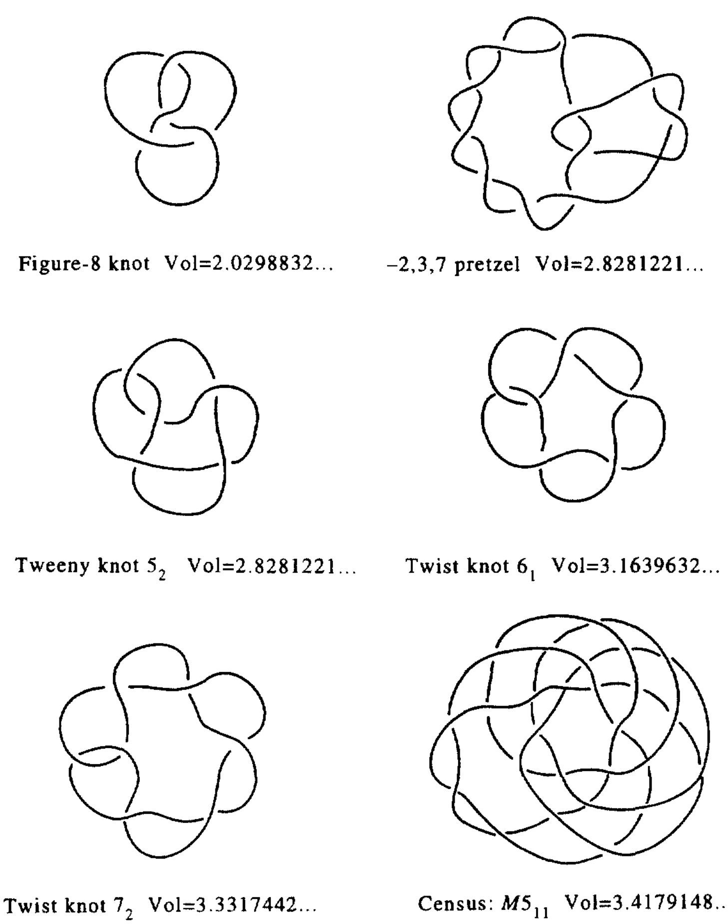


Fig. 6

Definition 5.2. By a hyperbolic knot we shall mean a knot K whose complement admits a hyperbolic structure of finite volume.

It may seem on first sight that the existence of a hyperbolic structure is a rather esoteric object. However, remarkably, it seems well-suited to provide the basis for many (computable) invariants to distinguish hyperbolic knots. Much of the remainder of the paper takes up this point. Before proceeding with this we make some observations on Theorem 5.1 to indicate that 'most knots are hyperbolic'. For instance for knots through 8 crossings all but four are hyperbolic, see Appendix 2 ([9] has lists through 10 crossings). Even more remarkable, is that from tables constructed by Hoste and Thistlethwaite [24], out of 313 230 prime knots through 15 crossings, only 21 are not hyperbolic. Some examples are given in Fig. 6 (calculation of volumes will be discussed below).

A specific corollary of Theorem 5.1 which establishes hyperbolicity for a large class of knots, and in particular for many of the knots through 10 crossings is the following established by Menasco [30].

Corollary 5.3. Let K be a prime alternating knot that is not a torus knot. Then K is hyperbolic. □

For extensions of this result to certain classes of non-alternating knots we refer the reader to [8] and [5].

6. THE SMITH CONJECTURE

An early (major) application of Thurston's Theorem 5.1 was to the solution to the Smith Conjecture. We give only but the briefest of accounts here; the book [38] contains the proof and has a detailed discussion of history and partial results.

Smith [60] proved in 1938 that any periodic (finite-order) orientation-preserving homeomorphism of S^3 to itself with fixed points has a fixed point set homeomorphic to the circle. He then asked if the circle must be unknotted. There are counter-examples [36] if the homeomorphism is not a diffeomorphism. The differentiable version of the question of unknottedness of the fix-point set is what became known as the Smith Conjecture.

6.1 Smith Conjecture

Let $h: S^3 \rightarrow S^3$ be an orientation-preserving, periodic diffeomorphism (different from the identity) with non-empty fixed-point set K. Then K is an unknotted circle.

In a tour de force involving the ideas of several mathematicians and much of the established (i.e. pre-Thurston) ideas of 3-manifold topology, Thurston's theorem became a catalyst for a complete proof.

Theorem 6.1. The Smith Conjecture is true. □

The proof proceeds by assuming for the sake of contradiction that K is knotted. It can be shown that if there is a counter-example to the Smith Conjecture there is one where K is a prime knot. The proof then naturally divides into two parts; namely when X_K does or does not contain a closed embedded incompressible surface of genus at least 1 which is not boundary parallel. When there is no such surface then Thurston's Theorem 5.1 applies (as discussed in Shalen's article in [38]) to give a contradiction. In the case of the existence of such a surface the article by Gordon and Litherland in [38] again establishes a contradiction. The work of Gordon and Litherland does not use hyperbolic structures, but crucially uses the so-called equivariant loop theorem due to Meeks and Yau (see their article in [38]) proved via the techniques of minimal surfaces.

A proof dependent only on hyperbolic structures is a consequence of the Orbifold Conjecture (see [26]) which is still to be resolved (see also the discussion in Section 10.2).

7. IDEAL TRIANGULATIONS

Thurston's Theorem 5.1 provides the existence of hyperbolic structures on a large class of knot complements. A natural question is how best to visualize, work with, and use this (unique) hyperbolic structure. We now discuss another beautiful idea due to Thurston [62] which illuminates the construction of a hyperbolic structure. We begin by expanding on the discussion of ideal tetrahedra given in Section 4.2.

7.1

Recall from Section 4.2 that an ideal tetrahedron is a tetrahedron in $\mathbb{H}^3 \cup S^2_\infty$ all of whose vertices lie on the sphere-at-infinity, and whose edges are hyperbolic geodesics. If T is an ideal tetrahedron, then as discussed in Section 4.2, T is determined up to isometry by the dihedral angles α, β and γ or, equivalently by the Euclidean triangle cut out by a horospherical cross-section at a vertex. However, as Thurston describes in Chapter 4 of [62] this triangle, and hence the ideal tetrahedron is essentially completely determined by a single complex number with positive imaginary part. We expand on this briefly below following the accounts in [62] and [48].

Let $\Delta = \Delta(u,v,w)$ be a Euclidean triangle in the complex plane with vertices u, v and w labelled counter-clockwise. To each vertex of Δ we associate the ratio of the sides incident at that vertex (see Fig. 7)

$$z(u) = \frac{w-u}{v-u},\ z(v) = \frac{u-v}{w-v},\ z(w) = \frac{v-w}{u-w}.$$

These vertex invariants depend only on the (orientation-preserving) similarity class of the triangle Δ. Notice that if we use an element of PSL(2,$\mathbb{C}$) to locate the vertices v and w at 0 and 1 respectively, then the third vertex is located at $z(u)$, and from this and the discussion above, it follows that $z(u)$ completely determines $z(v)$ and $z(w)$. Explicitly

$$z(v) = \frac{1}{1-z(u)} \text{ and } z(w) = \frac{z(u)-1}{z(u)}.$$

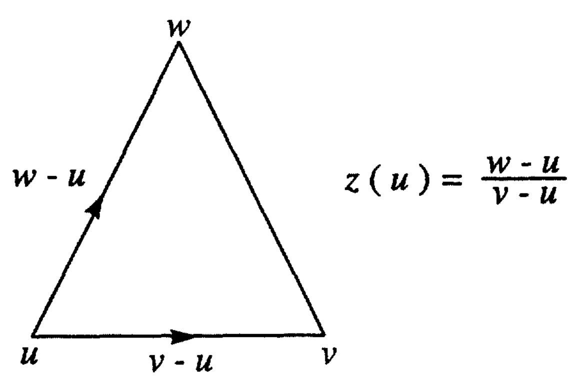


Fig. 7

Furthermore $z(u)$ has positive imaginary part and $\arg(z(u))$ is the angle at the vertex u. Using this description allows the following parametrization of Euclidean triangles which summarizes the above discussion.

Theorem 7.1. Let $\Delta = \Delta(u,v,w)$ *be a Euclidean triangle in the complex plane with vertices* u, v *and* w *labelled counter-clockwise, and let* $z_1 = z(u)$, $z_2 = z(v)$ *and* $z_3 = z(w)$ *be its vertex invariants. Then* z_1, z_2 *and* z_3 *have positive imaginary parts and satisfy the equations*

1. $z_1 z_2 z_3 = -1$, *and*
2. $1 - z_2 + z_1 z_2 = 0$.

Conversely, if z_1 z_2 *and* z_3 *are complex numbers with positive imaginary parts satisfying* (1) *and* (2) *above, then there is a Euclidean triangle* Δ *in* $\mathbb{C}$ *that is unique up to orientation-preserving similarity whose vertex invariants in counter-clockwise order are* z_1, z_2 *and* z_3. □

With this theorem, our previous discussion on ideal tetrahedra can be succinctly summarized in the following. First, observe that if T is an ideal tetrahedron in $\mathbb{H}^3$ and v is a vertex we can label the edges of T incident at v with the vertex invariants z_1, z_2 and z_3 of the Euclidean triangle $L(v)$ described above. Then opposite edges of T have the same labels. The complex numbers z_1, z_2 and z_3 are called the *edge invariants* of T. Summarizing we have:

Theorem 7.2. Let z_1, z_2 *and* z_3 *be complex numbers with positive imaginary part satisfying* (1) *and* (2) *of Theorem* 7.1. *Then, there is an ideal tetrahedron* T *in* $\mathbb{H}^3$ *unique up to orientation-preserving isometry, whose edge invariants are* z_1, z_2 *and* z_3. □

Furthermore, as noted above the complex numbers z_2 and z_3 are given in terms of z_1. Therefore, with a slight ambiguity, the tetrahedron T is completely determined by a single complex number z, which we call the *tetrahedral parameter* of T. The ambiguity is removed by fixing an edge and associating z as its edge parameter; the other edge parameters are then $1/(z-1)$ and $1-(1/z)$.

7.2

Thurston's idea to give a concrete description of hyperbolic structures on certain 3-manifolds was simply to give necessary and sufficient conditions so that gluing a finite collection of ideal tetrahedra together resulted in a 3-manifold admitting a hyperbolic structure of finite volume.

Thus suppose we have a 3-manifold M obtained by gluing ideal hyperbolic tetrahedra $T_1,\ldots, T_n$ by hyperbolic isometries, and we have fixed, as discussed above, an edge in each T_i and labelled this edge by the tetrahedral parameter z_i. We wish to decide conditions guaranteeing the z_i correspond to a complete hyperbolic structure of finite volume on M.

The decomposition of M into these ideal tetrahedra determines a not necessarily complete hyperbolic structure on $M\backslash$1-skeleton which we wish to extend to M. This is done by considering the image of the developing map (recall Section 3.1) in a neighborhood of an edge. Briefly, consider an edge E in M, lift to $\mathbb{H}^3$, and take a horospherical cross-section determined by a horosphere $\mathcal{H}$ centered at an ideal vertex v at an endpoint of E. The existence of a hyperbolic structure (not necessarily complete) forces triangles cut out of adjacent tetrahedra to E to line up 'neatly' around E—see Fig. 8. Each

$$z(e_i) \in \left\{ z_i, \frac{1}{z_i - 1}, 1 - \frac{1}{z_i} \right\}.$$

This condition is known as the *consistency condition* for a hyperbolic structure and is given algebraically by:

$$z(e_1)z(e_2)\ldots z(e_m) = 1 \text{ and } \arg z(e_1) + \ldots \arg z(e_m) = 2\pi$$

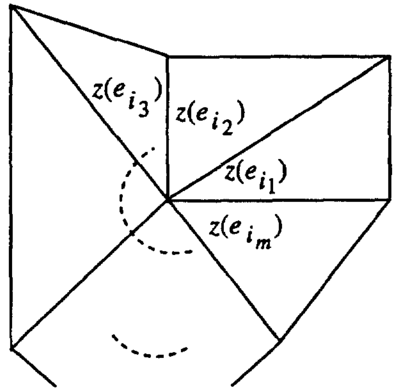


Fig. 8

To exhibit *completeness*, we note first that it is a consequence of the discussion in Section 3.2 that near an ideal vertex a complete hyperbolic structure of finite volume has the form (horoball)/$\mathbb{Z} \oplus \mathbb{Z}$. Therefore, we must ensure that the developing map near the ideal vertex v yields a Euclidean structure on the horosphere $\mathscr{H}$. This is seen by ensuring the developing map yields a tesselation of $\mathscr{H}$ by Euclidean triangles similar to those with parameters $z_1,\ldots, z_n$. To clarify this discussion we give some examples below.

7.3 Example 1: The figure-eight knot (see Fig. 9 (a))

The complement of the figure-eight knot admits a decomposition into two ideal tetrahedra, both isometric to $\mathscr{T}_0$. The tetrahedral parameter of $\mathscr{T}_0$ is $\frac{(1+\sqrt{-3})}{2}$. A horospherical cross-section is shown below. This yields the volume shown in Fig. 6.

Using this decomposition into two tetrahedra both isometric to $\mathscr{T}_0$, the volume of the figure-eight knot complement can be computed as in Section 4.2. By Theorem 4.3 we deduce that this volume is $2v_0$ (which gives the volume shown in Fig. 6).

The two examples below show horospherical cross-sections of a decomposition of the complements of the 5_2 knot and $(-2,3,7)$-pretzel knots (recall Fig. 6) into ideal tetrahedra. The first is a decomposition into 4 tetrahedra and the second into 3. The tetrahedral parameters of the tetrahedra are listed below. Volumes can computed from this using Section 4.2.

7.4 Example 2: The knot 5_2 (see Fig. 9(b))

Each tetrahedron is similar to one where the tetrahedral parameter is a root of either $x^3-x+1=0$ or $x^3-5x^2+4x-1=0$. The approximate values of the roots are $0.66235897862+0.56227951206i$ or $0.460202188254+0.182582254557i$ respectively. These roots generate the same cubic extension.

7.5 Example 3: The $(-2,3,7)$ *– pretzel knot* (see Fig. 9(c))

The tetrahedral parameters in this case are all given by a root of $x^3-x+1=0$.

This very concrete realization of the hyperbolic structures on knot complements is most useful. One reason, as we have seen, is in computation of volumes, and another is provided by the following theorem due to Epstein and Penner [17].

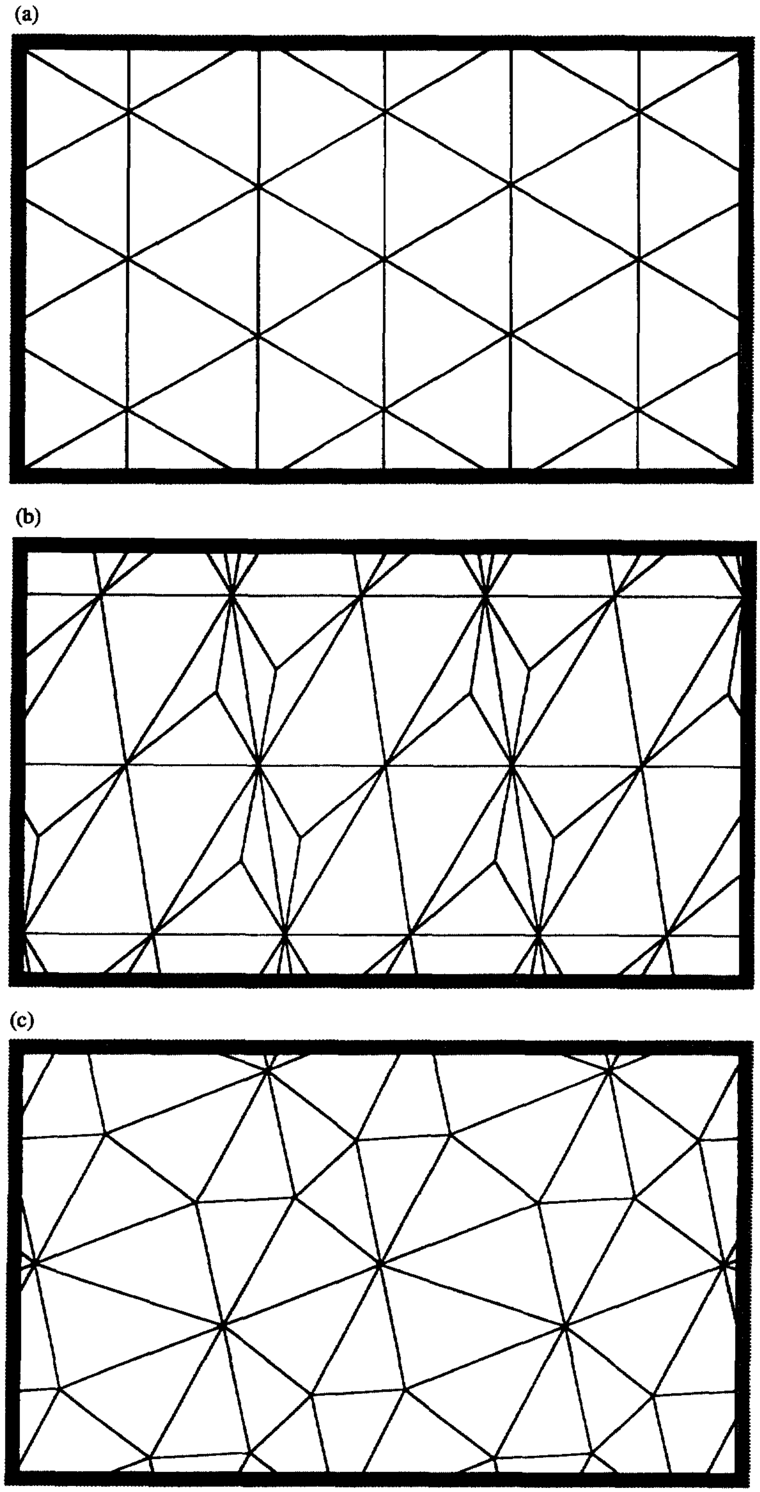


Fig. 9. (a) Ideal triangulation of the figure-eight knot complement.(b) Ideal triangulation of the tweeny knot complement.(c) Ideal triangulation of the (-2,3,7)-knot complement.

Theorem 7.3. Every non-compact finite volume hyperbolic 3-manifold admits a decomposition into a finite number of ideal polyhedra. □

It is conjectured that this subdivision can be further subdivided to give a decomposition into ideal tetrahedra. Indeed, in all known examples this is the case, see [66] and [59] for more on this.

For hyperbolic knot complements, following ideas Thurston used for the figure-eight, there are many instances where an ideal triangulation can be explicitly built from a projection of the knot. We briefly describe the heuristics on how this is done; for a fuller discussion of this see [1,31,46].

Let K be a hyperbolic knot, P a projection plane in S^3 (recall Section 2.1) for K, and $G(K)$ the graph on P which is the projection of K (we assume the projection is regular). The graph $G(K)$ induces a cell decomposition of S^3; the 0-cells being the crossings, the 1-cells are the edges of $G(K)$, the 2-cells being the complementary regions of $P \setminus G(K)$ and the 3-cells the components of $S^3 \setminus P$ which are simply two 3-balls. Now modify this cell decomposition of S^3 at each vertex of $G(K)$ as follows. At each vertex of $G(K)$ there is an overpassing arc and an underpassing arc. Push the overcrossing arc above P and the undercrossing arc below P, and add a short vertical segment joining the branches. This gives a new complex, where the 0-cells are the endpoints of these short segments, the 1-cells consist of edges of $G(K)$ and the additional segments, the 2-cells are the complementary regions as above which are modified near the crossings of $S^3 \setminus P$ by sewing in appropriate twisted discs, and the 3-cells are seen to be a pair of open 3-balls. It is from this basic cell decomposition that a decomposition of the complement into tetrahedra can often be done.

Using Theorem 7.3 as a starting point, it is possible to modify the decomposition provided by Theorem 7.3 to produce a *canonical* decomposition of any non-compact hyperbolic 3-manifold of finite volume into a finite number of ideal hyperbolic *polyhedra*. By canonical we mean that the decomposition depends only on the geometry of the manifold, and not on the ideal decomposition used as input data. We will not go into the details of this construction here, see [66] for details. Moreover, this construction is entirely algorithmic which makes it most useful in practical applications. In particular the computer program *SnapPea* due to Jeff Weeks [67] has implemented this algorithm. Using SnapPea it is remarkable that canonical decompositions can then be computed from knot diagrams of up to around 50 crossings in just a few seconds. Once a canonical cell decomposition is known, the combinatorial structure completely determines the topological type of the manifold, and hence by the Rigidity theorem of Mostow and Prasad two such manifolds (and hence two hyperbolic knots) are isometric (resp. equivalent) if and only if they have the same canonical cell decompositions.

The canonical decomposition also has applications in computing symmetry groups of hyperbolic knot complements as we will discuss below in Section 8.

Remark. It follows from the Rigidity theorem of Mostow and Prasad that for the complete structure the tetrahedral parameters of any tetrahedra occurring in a decomposition of a non-compact finite volume hyperbolic 3-manifold are algebraic numbers. For more on related topics, see Section 10.

7.6

The existence of an ideal decomposition of a non-compact finite volume hyperbolic 3-manifold into ideal tetrahedra also allowed Thurston to exhibit hyperbolic structures on closed 3-manifolds by *hyperbolic Dehn surgery*. We briefly describe this for hyperbolic knots in S^3, see [62,12,41] for more details.

Let K be a hyperbolic knot with a decomposition into ideal tetrahedra $T_1,\ldots, T_n$ as above. As described above, one can equip $S^3\backslash K$ with many *incomplete* hyperbolic structures by varying the shapes of the T_i. These are paramaterized by tetrahedral parameters satisfying the consistency conditions above (cf. Section 7.2). Thurston [62] proved that the metric completion of many of these incomplete structures on $S^3\backslash K$ give rise to complete hyperbolic structures on certain closed 3-manifolds. This procedure is called *hyperbolic Dehn surgery.* Moreover the closed manifolds turn out to be those obtained by *topological Dehn surgery* on K. By this we mean the following. Let α be an essential simple closed curve on ∂X_K, V be a solid torus and r a meridional curve of V. Now glue V to X_K along their boundaries so that α is identified with r. The result is a closed 3-manifold obtained by α-Dehn surgery on K. From Section 2.1, upon specifying a framing $\{\mathcal{M},\ell\}$, α can be described as $\mathcal{M}_p\ell^q$ and α-Dehn surgery is referred to as (p,q)-Dehn surgery and denoted $X_K(p,q)$. For instance, $X_K(1,0)=S^3$ for any knot K. In the notation just introduced, Thurston's Hyperbolic Dehn Surgery Theorem says,

Theorem 7.4. Let K be a hyperbolic knot. For all but finitely many (p,q), $X_K(p,q)$ is a closed hyperbolic 3-manifold. □

Thurston also shows that for those (p,q) which yield hyperbolic Dehn surgeries, $\mathrm{Vol}(X_K(p,q)) < \mathrm{Vol}(S^3\backslash K)$ and the volumes $\mathrm{Vol}(X_K(p,q))$ accumulate on $\mathrm{Vol}(S^3\backslash K)$.

The structure of the set of volumes of hyperbolic 3-manifolds is itself very interesting. We will not discuss this here; we refer the reader to [62] and the article by Gromov [22] for more information. The only facts we will have recourse to are included in:

Theorem 7.5. There is a lower bound to the volume of a hyperbolic 3-manifold. Furthermore, there are only finitely many hyperbolic 3-manifolds of the same volume. □

An immediate corollary of the existence of this lower bound together with the fact that under a covering of degree d volume multiplies by a factor of d, is that a hyperbolic 3-manifold of finite volume covers at most finitely many hyperbolic 3-manifolds (indeed orbifolds).

Remarks.

1. Describing exactly what non-hyperbolic Dehn surgeries can be obtained by Dehn surgery on a hyperbolic knot is currently an important question in 3-manifold topology. We refer the reader to [20] for a survey of some recent results.
2. By definition, the fundamental groups of those closed hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on a hyperbolic knot K admit a holonomy representation into $\mathrm{SL}(2,\mathbb{C})$. As these manifolds are the result of topological Dehn surgery, the fundamental group of $S^3\backslash K$ subjects the groups $\pi_1(X_K(p,q))$. By composing this map with the holonomy representation, we obtain many more homomorphisms of $\pi_1(S^3\backslash K)$ into $\mathrm{SL}(2,\mathbb{C})$. These yield points in the *Representation Variety* of $\pi_1(S^3\backslash K)$ and this algebraic object turns out to have a powerful influence on the topology of X_K. For more on this topic see the article by Cooper and Long in this issue [16].

8. SYMMETRIES

A topic that has been the subject of much attention in classical knot theory is identifying symmetries of a knot (see the references in [14] for some of this work). A *symmetry* of a knot K is a homeomorphism of the pair (S^3,K) to itself, and the collection of symmetries forms a group $\mathrm{Symm}(K)$.

Equipping S^3 and K with orientations, we have:

- Suppose $h:(S^3,K) \to (S^3,K)$ preserves the orientation of S^3 but reverses the orientation of K, then K is called *invertible.* Such a symmetry is called an *inversion.*
- Suppose $h:(S^3,K) \to (S^3,K)$ reverses the orientation of S^3 then K is called *amphicheiral.* There are two types of amphicheiral symmetries: if h preserves the orientation of K then h is called $+$*amphicheiral.* If h reverses the orientation of K then h is called $-$ *amphicheiral.* A knot with both types of orientation-reversing symmetries is called $\pm$ *amphicheiral.*

If we fix a framing $\{\mathcal{M},\ell\}$, it is not hard to check that K is invertible (resp. amphicheiral) if and only if there is a homeomorphism ϕ of X_K such that $\phi(\mathcal{M}) = \mathcal{M}^{-1}$ and $\phi(\ell) = \ell^{-1}$ (resp. $\phi(\mathcal{M}) = \mathcal{M}^{-1}$ and $\phi(\ell) = \ell$).

Consider the diagram of the figure-eight knot below (Fig. 10).

A 180 degree rotation about the y-axis takes the knot back to itself preserving both the orientation of S^3 and the knot. A 180 degree rotation about the x-axis preserves the orientation of S^3 but reverses the orientation of the knot (note that this axis hits the knot in two places, this is called a *strong inversion*). There is also an orientation-reversing symmetry of S^3 which sends the knot back to itself. This involution preserves the orientation of the knot, hence it is *amphicheiral.* If we compose this symmetry with the strong inversion mentioned above we get an orientation-reversing symmetry of S^3 which reverses the orientation of the knot (hence it is $-$*amphicheiral*). Thus the figure-eight knot possesses all four types of symmetry.

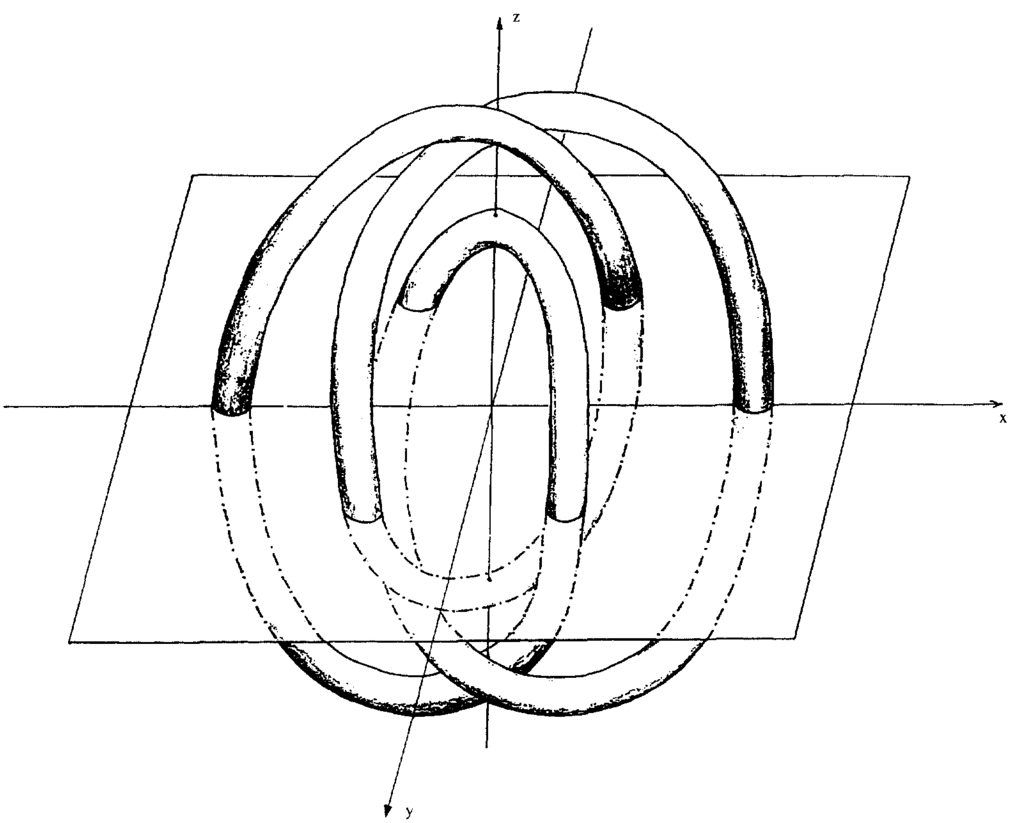


Fig. 10

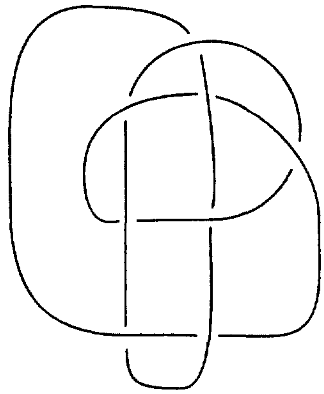


Fig. 11

Finding symmetries can be easy if one is presented a 'nice' symmetric diagram of the knot. However, it is quite difficult in general to find all symmetries of a knot, or prove there are no symmetries. As evidence of this, it was not until 1963 that Trotter [64] gave the first example of a knot that was not invertible. This knot turns out to be the knot 8_{17} in the tables of [57] (it is pictured in Fig. 11).

For hyperbolic knots, the Rigidity theorem of Mostow and Prasad coupled with work of Waldhausen [65] implies that Symm(K) is isomorphic to the group of isometries of the knot complement Isom($S^3\backslash K$) which additionally is a finite group, see [62] Chapter 5. Let $\mathrm{Isom}_+(S^3\backslash K)$ denote the group of orientation-preserving isometries of $S^3\backslash K$. Then $\mathrm{Isom}_+(S^3\backslash K)$ is a subgroup of the index at most 2 in Isom($S^3\backslash K$), the index being 2 exactly when K is amphicheiral. If Γ_K denotes the Kleinian group obtained as the image of the holonomy representation of $\pi_1(S^3\backslash K)$ into PSL(2,$\mathbb{C}$), and Norm(K), the normalizer of Γ_K in Isom($\mathbb{H}^3$) (resp. $\mathrm{Norm}_+(K)$ is the normalizer of Γ_K in PSL(2,$\mathbb{C}$)), then the group Isom($S^3\backslash K$) is isomorphic to the finite quotient group Norm(K)/Γ_K (recall Theorem 3.5). The solution to the Smith Conjecture (Theorem 6.1) provides the following restrictions on Symm(K) (cf. the discussion in [55] or [27]).

Theorem 8.1. Let K be a hyperbolic knot. Then $\mathit{Isom}_+(S^3\backslash K)$ *is either the dihedral group* D_n *of order* $2n$*, or a cyclic group of order n.* $\mathit{Isom}(S^3\backslash K)$ *is at most a* $\mathbb{Z}_2$*-extension of* $\mathit{Isom}_+(S^3\backslash K)$. □

For hyperbolic knots, the canonical decomposition into ideal polyhedra can be used to calculate the group of symmetries of K. In this case the symmetry group of the knot is the same as the symmetry group of the canonical decomposition. These combinatorial symmetries can then be checked to see how they act on ∂X_K so as to determine the exact type of each symmetry. Henry and Weeks [23] implemented a routine into *SnapPea* for calculating the symmetry groups via this method. They used this in [23], to calculate the symmetry groups of all the knots in the tables of [57], and verify the independent work of [27]. We briefly discuss some examples (see Table 1 of Appendix B).

In Example 1 of Section 7.2, a decomposition of the figure-eight knot complement into two ideal tetrahedra was described. This is in fact the canonical decomposition, and it can be quickly deduced from this that the full symmetry group of the figure-eight knot is D_4 the dihedral group of order 8.

Examples 2 and 3 in Section 7.2 describe decompositions of the complements of the knot 5_2 and the $(-2,3,7)$-pretzel knot into 4 and 3 ideal tetrahedra respectively. Again this is the canonical decomposition, and the symmetry groups can be shown to be $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_2$ respectively. Both knots are non-amphicheiral.

The knot 8_{17} (shown below) is the first non-invertible knot in the tables of [57].

It has a canonical decomposition into 16 ideal tetrahedra. One can then check using SnapPea that $\mathrm{Isom}(S^3\backslash 8_{17}) = \mathbb{Z}_2$ and the non-trivial symmetry reverses the orientation of both the knot and S^3 so 8_{17} is $-$amphicheiral. This agrees with Trotter's calculation discussed above.

Some further remarks concerning detection of symmetries via the existence of a hyperbolic structure are discussed below in Section 10.1. Table 1 of Appendix B at the end of this paper includes a list of the symmetry groups of hyperbolic knots with small numbers of crossings.

9. INVARIANTS ASSOCIATED WITH THE HYPERBOLIC STRUCTURE

In this section we discuss applications of the existence of a hyperbolic structure to knot theory.

9.1

The problem of determining when knots are or are not equivalent has long occupied knot theorists. By Theorem 2.2, knots are determined by their complements, which in the context of hyperbolic knots, together with the Rigidity theorem of Mostow and Prasad mentioned in Section 3.1, shows that hyperbolic knots are equivalent if and only if their complements are isometric as hyperbolic 3-manifolds, or that the fundamental groups as Kleinian groups are conjugate in $\mathrm{Isom}\mathbb{H}^3$. For instance if K_1 and K_2 are hyperbolic knots with $\mathrm{Vol}(S^3\backslash K_1) \neq \mathrm{Vol}(S^3\backslash K_2)$ then K_1 and K_2 are not equivalent. By Theorem 7.5 there can only be finitely many hyperbolic knots whose complements have the same volume and so volume is a most useful tool in distinguishing knots—especially in view of the decomposition of hyperbolic knot complements into ideal tetrahedra. It is worth comparing this with the fact that there are infinitely many knots with the same Alexander polynomial [39] or HOMFLY polynomial [25].

However, there are many knots whose complements have the same volume. These are produced by a well-known construction in knot theory called *mutation*, which produces knots that tend to be hard to distinguish. As we now discuss, hyperbolic volume is of no use here (although we will see that more subtle invariants of the hyperbolic structure can be). Informally, knots K_1 and K_2 are mutants of one another if a diagram for K_2 can be obtained from one of K_1 by cutting out a tangle (shown below as the interior of the dotted circle) and then sewing it back in via a rotation through π.

To define this more carefully, we require some notation.

Definition 9.1. Let K be a knot. A Conway sphere for K is an embedded 2-sphere in S^3 meeting the knot transversely in four points.

Let K be a knot, S a Conway sphere for K. Write $(S^3,K) = (B^3_+,K_+) \cup (B^3_-,K_-)$ where $B^3_\pm$ are the 3-balls bounded by S, and $K_\pm = B^3_\pm \cap K$. Any sphere with four punctures admits three obvious symmetries which are rotations through π. Let μ be any of these three involutions. The *mutation* of K via μ is obtained by cutting open $S^3\backslash K$ along S and then regluing via μ. Thus,

$$(S^3,K^\mu) = (B^3_+,K_+) \cup_\mu (B^3_-,K_-)$$

It is known that mutants have the same abelian invariants, and same Jones polynomial (see [61,28]). The following theorem was proved by Ruberman [58], giving the hyperbolic volume version.

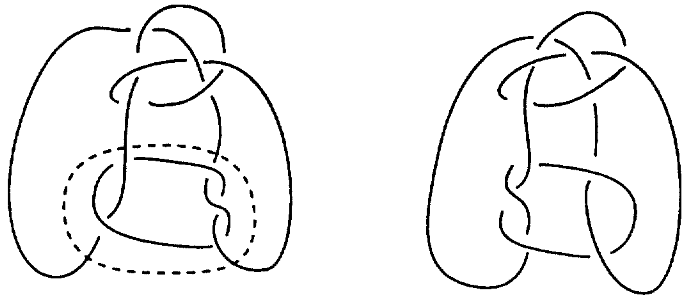


Fig. 12

Theorem 9.2. Let K be a hyperbolic knot, S an incompressible Conway sphere and μ an involution of S as above. Then K^{μ} is hyperbolic and $Vol(S^3\backslash K) = Vol(S^3\backslash K^{\mu})$. □

Figure 12 shows a pair of famous mutants due to Conway (on the left) and Kinoshita–Terasaka. The common volume is approximately 11.219117725.... However, the canonical cell decompositions of the complements of the mutant knots shown in Fig. 12 are comprised of 14 and 15 ideal tetrahedra respectively, and so from the discussion in Section 7.2 are definitely not equivalent. These were first shown to be distinct by Riley [51], and later Gabai [18], who explicitly computed their genus which also showed them to be distinct.

Remark. By Theorem 7.5 there are only finitely many hyperbolic knot complements of the same volume. One can show using sequences of mutations on Montesinos knots (see [14]) that for any positive integer N there are N distinct hyperbolic knots whose complements have the same volume. The crucial feature about Montesinos knots is that their classification is very explicit, see [14].

9.2

There are several obvious questions that arise at this point. Can the volumes of hyperbolic knot complements be made arbitrarily large? What is the smallest volume of a hyperbolic knot complement? Recall Theorem 7.5 states that there is a lower bound to the volume of a hyperbolic 3-manifold. The answer to the first question is yes, but the answer to second question is not yet known.

Conjecture. The figure-eight knot complement has the smallest volume of any hyperbolic knot complement.

As stated in Example 1 of Section 7, the volume of the figure-eight knot complement is $2v_0$. This volume is in fact conjectured to be the smallest volume of *any* non-compact finite volume hyperbolic 3-manifold.

Theorem 9.3. The volumes of hyperbolic knot complements can be made arbitrarily large.

Sketch Proof. Recall the notion of a framing from Section 2.1. Consider an unknot in S^3 with the ‘obvious’ framing. The effect of $(1,n)$-Dehn surgery (recall Section 7.3) on this unknot in S^3 is to give back S^3. Now consider a link in S^3 for which some part of a projection has the form shown in Fig. 13(a). If we fix a framing of this unknotted component (as above) and perform $(1,n)$-Dehn surgery on this unknotted component, we obtain a link in S^3 (with 1 fewer components) and the effect on the two strands is to twist them together as shown in Fig. 13(a).

Now consider the family of links shown in Fig. 13(b). It is proved in [1] that these links are hyperbolic with volume approximately $(m-1)(7.32772...)$. On choosing a framing for each unknotted component $L_1,...,L_m$ of $\mathscr{L}_m$ we can perform $(1,n)$ Dehn surgery on each of the

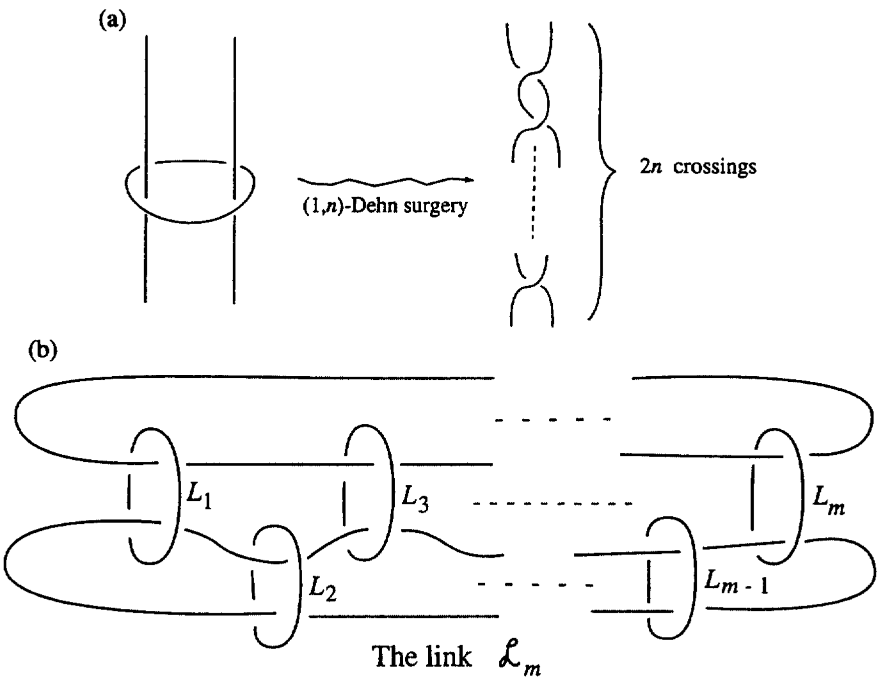


Fig. 13

unknotted components $L_1,\ldots,L_m$ of $\mathcal{L}_m$. For suitable n, this produces a family of knots which are hyperbolic and Thurston's Hyperbolic Dehn Surgery Theorem in this context says that the volume of these knot complements approach the volume of the manifolds $S^3\backslash\mathcal{L}_m$. Since these volumes are getting arbitrarily large the result follows. □

9.3

Classical measures of complexity of a knot include the *crossing number*, the *bridge number* and the *genus* of a knot (recall Section 2.3).

However the crossing number of a knot does not in general reflect the geometric complexity of the complement. For example, for fixed m, the family of knots shown in Fig. 13(a) can have arbitrarily large crossing number, but have volume bounded by $(m-1)(7.32772\ldots)$. The number of tetrahedra in the canonical decomposition also increases as the number of crossings increase, and so one might be tempted to look for a correlation between the crossing number of a knot and the number of tetrahedra in the canonical decomposition of its complement, but this also does not tend to be true. The knot shown in Fig. 14 has crossing number 23, yet its complement has volume approximately 3.6086890618... and decomposes into 4 ideal tetrahedra (it appears as $M4_{31}$ in the census of finite volume non-compact hyperbolic 3-manifolds [15]).

There is however an upper bound for the volume of a hyperbolic knot in terms of the crossing number due to Adams [1]. This arises from producing an ideal decomposition for the knot complement that is built from a diagram of the knot (recall Section 7.2), and then apply Theorem 4.3.

Theorem 9.4. Let K be a hyperbolic knot of crossing number m, different from the figure-eight knot. Then $Vol(S^3\backslash K) \leq (4m-16)v_0$. □

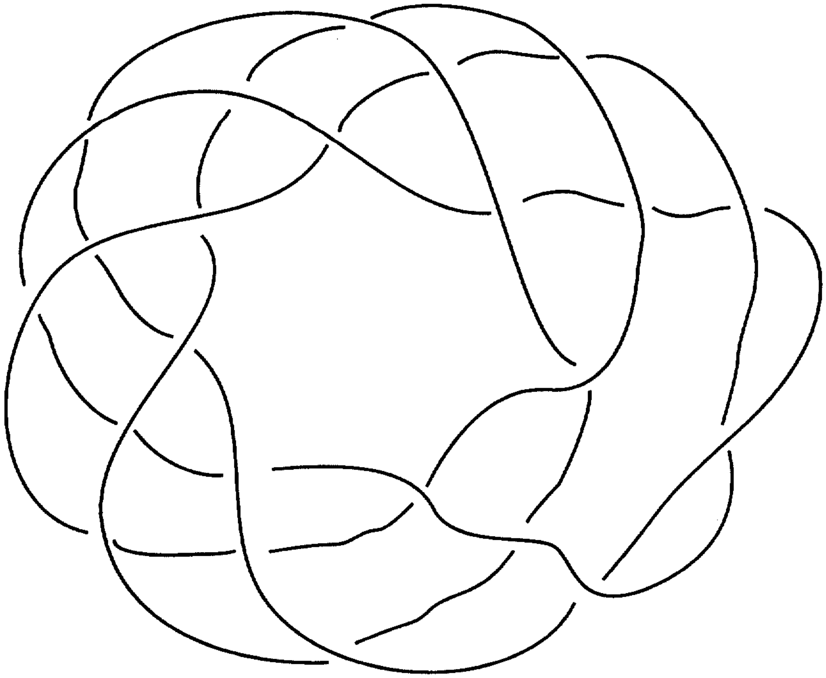


Fig. 14

The knot in Fig. 14 is also interesting because, despite its small volume it is of genus 10. The (− 2,3,7)-pretzel knot has genus 5 and so it would appear that genus does not seem to have much correlation to the complexity of the hyperbolic structure.

Bridge number and more particularly *tunnel number* do seem to be much more closely related to the complexity of the hyperbolic structure on a hyperbolic knot complement. Roughly speaking this is because the fundamental group is getting more complicated as these invariants increase. The tunnel number of a knot K is the minimal number of properly embedded arcs in X_K such that the complement of an open regular neighborhood of the arcs is a handlebody. Such a collection of arcs are called *unknotting tunnels* for K. Figure 15 shows an unknotting tunnel for the figure-eight knot.

It is not hard to see that the tunnel number of $K \leq$ (the bridge number of $K - 1$). We refer the reader to [6,10] for much more on the connections of tunnel number to the complexity of

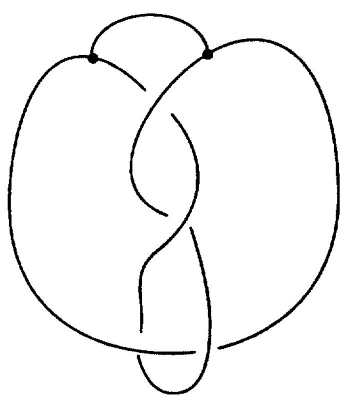


Fig. 15

the hyperbolic structure. We simply quote the following results contained in [6] that also gives some partial evidence to the conjecture that the figure-eight knot 'is the smallest hyperbolic knot'. Recall the figure-eight knot complement has volume $2v_0$.

Theorem 9.5. There exists a universal constant C such that if K is a hyperbolic knot with tunnel number t, then $Vol(S^3\backslash K) \geq Ct$. □

Theorem 9.6. Let K be a hyperbolic knot of tunnel number at least 2. *Then* $Vol(S^3\backslash K)$ *is at least* $3v_0/2$. □

9.4

In light of the conjecture that the figure-eight knot complement is the smallest hyperbolic knot, it is natural to try estimate the volume of a hyperbolic knot (we have already seen some results in this direction). Here we discuss this further.

We begin by discussing a method of obtaining a lower bound on the volume originating in work of Meyerhoff [32] and pursued by Adams [2] and [3]. We will discuss their work only in the context of hyperbolic knots, but the applications are more widespread.

If K is a hyperbolic knot, the knot complement has a topological end of the form $T^2 \times [0,\infty)$, called a *cusp*. The knot complement can then be informally viewed as a 'thick' part together with a cusp. To estimate the volume of $S^3\backslash K$ one can try to estimate the volume of a cusp. More precisely, lifting to $\mathbb{H}^3$ the cusp is covered by an infinite set of disjoint horoballs in $\mathbb{H}^3$ all of which are identified under covering transformations. Expand these horoballs equivariantly until two first touch. Projecting to $S^3\backslash K$ gives a *maximal cusp*, and the volume of this maximizes the volume of a cusp, and hence the volume of a maximal cusp is what could be computed to give a lower bound to the volume of $S^3\backslash K$. This discussion is valid for any non-compact finite volume hyperbolic 3-manifold. We have the following from [3]:

Lemma 9.7. Let M be a finite volume non-compact hyperbolic 3-manifolds and C a maximal cusp for M. Then $Vol(C) \geq \sqrt{3}/2$. □

This result can be used directly to estimate the volume of a hyperbolic knot complement, however, using sphere packing arguments due to Boroczky [13] and work in [32], a rather better estimate can be made, see [3].

Theorem 9.8. Let K be a hyperbolic knot, then $Vol(S^3\backslash K) \geq v_0$. □

We have been focused only on orientable manifolds, if we allow non-orientable hyperbolic 3-manifolds of finite volume, then [2] identifies the smallest non-orientable hyperbolic 3-manifold of finite volume. It is the *Gieseking manifold* and is obtained by identifying the faces of $\mathcal{T}_0$ in pairs. It has volume v_0 and is double covered by the figure-eight knot complement. This result together with Theorem 9.4 allows one to estimate the orders of fixed-point free symmetries of hyperbolic knots. Namely,

Corollary 9.9. Let K be a hyperbolic knot with m crossings different from the figure-eight knot. If g is a fixed-point free symmetry of $S^3\backslash K$ *then g has order at most* $4m - 16$.

Proof. By Theorem 9.4 the volume of $S^3\backslash K$ is at most $(4m-16)v_0$. If we denote by I_g the subgroup of $\mathrm{Isom}(S^3\backslash K)$ generated by g, then $(S^3\backslash K)/I_g$ is a finite volume non-compact hyperbolic 3-manifold, which may be non-orientable. However, as discussed the minimum volume of any non-compact hyperbolic 3-manifold is v_0, and so the corollary now follows as the order of I_g is then bounded above by $(4m-16)v_0/v_0$. □

There are also similar applications where the isometry has fixed points using the results of [33] which identifies the minimal volume non-compact hyperbolic 3-orbifold (recall Section 3.1), see also [4].

Indeed there are constraints on the possible orders of homeomorphisms acting fixed-point freely simply from the volume of a maximal cusp. Using Lemma 9.7 and arguing similarly to the proof of Corollary 9.9 the following can be established [3].

Corollary 9.10. Let K be a hyperbolic knot and g a fixed-point free orientation-preserving symmetry of $S^3 \backslash K$, *then g has order at most* $Vol(C)2/\sqrt{3}$ *where C is a maximal cusp for* $S^3 \backslash K$. □

The horoballs described above are centered at parabolic fix-points of the elements in the holonomy representation of $\pi_1(S^3 \backslash K)$ into SL(2,$\mathbb{C}$). By locating one of these fix-points at infinity, we produce an invariant of the knot called the *horoball diagram* for K. Although hard to work with directly, SnapPea computes a portion of the horoball diagram by specifying a radius of the smallest horoball to be computed and this can be used to distinguish knots. For instance the horoball diagrams of the mutant knots in Fig. 12 are included in Appendix A of this paper, and a quick inspection shows these are different.

Remark. Other geometric invariants that arise from the hyperbolic structure are the *Chern Simons* invariant, the *eta invariant* and *the length spectrum*. We will not discuss these invariants here—a discussion is given in [34].

10. ALGEBRAIC INVARIANTS ASSOCIATED WITH A HYPERBOLIC STRUCTURE

In this section we discuss algebraic properties related to the existence of a hyperbolic structure.

10.1

Suppose we have a decomposition of $S^3 \backslash K$ into ideal tetrahedra

$$S_1 \cup S_2 \cup \ldots \cup S_n$$

with tetrahedral parameters $z_1, z_2, \ldots, z_n$. Define the *field of tetrahedral parameters of K* denoted $k_\Delta(K)$ to be $\mathbb{Q}(z_i \mid i = 1, \ldots, n)$. This is a finite extension of $\mathbb{Q}$.

Some examples of this field computed from Section 7.2 are figure-eight knot, the 5_2 knot, and the $(-2,3,7)$-pretzel knot. These give $\mathbb{Q}(\sqrt{-3})$, and $\mathbb{Q}(z)$ where $z^3 - z^2 + 1 = 0$ respectively.

The existence of a hyperbolic structure provides the holonomy representation of $\pi_1(S^3 \backslash K)$ into PSL(2,$\mathbb{C}$), and thus an isomorphism of $\pi_1(S^3 \backslash K)$ onto a Kleinian group Γ_K. As noted in Section 3.2 this representation may be lifted to SL(2,$\mathbb{C}$). We can then define the *trace-field* of Γ_K to be the field $\mathbb{Q}(\mathrm{tr}\gamma \mid \gamma \in \Gamma_K)$. Since the trace of a 2×2 matrix is a conjugacy invariant, the Rigidity theorem of Mostow and Prasad implies that the trace-field is an invariant of the knot. The Rigidity theorem also implies that the trace-field is a finite extension of $\mathbb{Q}$. The following theorem is proved in [42] which allows one to establish that the tetrahedral parameters are algebraic numbers.

Theorem 10.1. Let K be a hyperbolic knot, then $\mathbb{Q}(tr\gamma \mid \gamma \in \Gamma_K) = \mathbb{Q}(z_i \mid i = 1, \ldots, n)$. □

We will abbreviate the 'trace-field of Γ_K' to simply the 'trace-field of K' and denote it by $\mathbb{Q}(\mathrm{tr}\Gamma_K)$. Theorem 10.1 has some interesting applications of which we discuss only a few.

It is extremely efficient at distinguishing knots. In Appendix B at the end of this paper we list the trace-field for knots up through 8 crossings. These were computed by Craig Hodgson.

Two Kleinian groups Γ_1 and Γ_2 are called *commensurable* if $\Gamma_1 \cap \Gamma_2$ have finite index in both of Γ_1 and Γ_2. It is not hard to see that two finite volume hyperbolic 3-manifolds $M_1 = \mathbb{H}^3/\Gamma_1$ and $M_2 = \mathbb{H}^3/\Gamma_2$ have a homeomorphic finite cover exactly when Γ_1 and some conjugate of Γ_2 are commensurable. The following proves *invariance* of the trace-field of a hyperbolic knot, see [49] and [42]:

Theorem 10.2. Let K be a hyperbolic knot, then $\mathbb{Q}(tr\gamma \mid \gamma \in \Gamma_K) = \mathbb{Q}(z_i \mid i = 1, \ldots, n)$ *is an invariant of the commensurability class of* $S^3 \backslash K$. □

Topologically deciding when manifolds have homeomorphic finite covers is in general quite hard. Theorem 10.2 is most useful given the existence of a hyperbolic structure. For instance, the knots in the table of Appendix B are all mutually non-commensurable. However note that the trace-field for the $(-2,3,7)$-pretzel knot and 5_2 knot are the same.

As discussed in Section 9.1 mutant knots are not distinguished by hyperbolic volume. Unfortunately, the trace-field is also mutation invariant [43].

Theorem 10.3. Let K be a hyperbolic knot and K^μ *a mutant of K. Then, K and* K^μ *have the same trace-field.* □

The trace-field also detects the properties of amphicheirality, and whether a knot admits a symmetry of order p with non-trivial fixed-point set. The next theorem is relatively easy to establish, see [55].

Theorem 10.4. Let K be a hyperbolic knot. Then,

- *if K is amphicheiral* $K = \bar{K}$ *and so contains real subfield of index* 2.
- *if K admits an orientation-preserving symmetry of order p with non-trivial fixed point set, then* $\mathbb{Q}(tr\Gamma_K)$ *contains the field* $\mathbb{Q}(\cos 2\pi/p)$. □

Table 1 of Appendix B can be used to quickly prove many knots up through crossings cannot be amphicheiral, and limit symmetries to order 2 or 3.

10.2

A particularly interesting class of hyperbolic 3-manifolds are the *arithmetic hyperbolic 3-manifolds.* If $M = \mathbb{H}^3/\Gamma$ is a non-compact finite volume hyperbolic 3-manifold, then M is arithmetic if some conjugate of Γ is commensurable with a group $PSL(2,O_d)$, where O_d is the ring of integers in the quadratic imaginary number field $\mathbb{Q}(\sqrt{-d})$. For instance Example 2 of Section 5 shows that the figure-eight knot complement is arithmetic. From this, and his other examples, Riley conjectured that the only 'arithmetic knot' was the figure-eight knot. From the list of fields in Appendix B of the paper it is easy to see that none of the knots through 8 crossings are arithmetic since if a knot is arithmetic the trace-field is a quadratic imaginary extension of $\mathbb{Q}$. Riley's conjecture was established in [50].

Theorem 10.5. Let K be a hyperbolic knot, and suppose that $S^3 \backslash K$ *is arithmetic. Then K is the figure-eight knot.* □

In particular Theorem 10.5 shows that the figure-eight knot complement is not commensurable with any other knot complement. There are examples of hyperbolic knot complements that admit a finite covering which again is a knot complement, for example the $(-2,3,7)$-pretzel knot has 18 and 19 fold cyclic covers which are knot complements in S^3 (for more on this topic see [19]). There are also a pair of examples due to Aitchison and Rubinstein [11], called the *dodecahedral knots* pictured in Fig. 16, which have commensurable hyperbolic complements, but are of the same volume and so cannot be covers of each other.

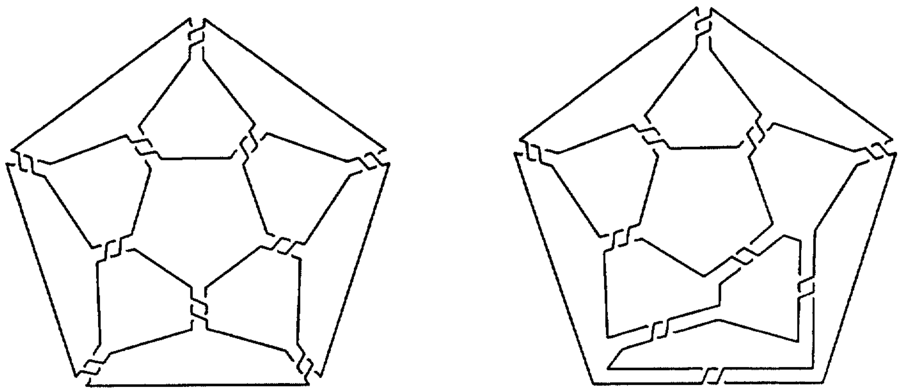


Fig. 16

In Section 8 we discussed symmetries of hyperbolic knot complements, an interesting extension of this is the notion of *hidden symmetry* of a knot complement. By a hidden symmetry of a hyperbolic knot K, we mean a symmetry of some finite covering of $S^3\backslash K$ that does not lift from $S^3\backslash K$. As discussed in Section 8, symmetries of the knot complement are related to the structure of the group Norm(K) (recall Section 8). Hidden symmetries are related to the structure of the *commensurability subgroup* of Γ_K. This defined as follows. Let Γ be a Kleinian group, define:

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{Isom}(\mathbb{H}^3) \mid g\Gamma g^{-1} \text{ is commensurable with } \Gamma\}.$$

In the case where $\Gamma = \Gamma_K$, we simply shorten the above to Comm(K). Note that if Γ is a Kleinian group then Norm(Γ) is a subgroup of Comm(Γ). It is a deep theorem of Margulis [29] that $M = \mathbb{H}^3/\Gamma$ is arithmetic if and only if Comm(Γ) is dense in Isom($\mathbb{H}^3$), and otherwise Comm(Γ) is discrete and is the unique maximal group in the commensurability class of Γ. By Theorem 10.5, it is not hard to see that this means the figure-eight knot has *infinitely many* hidden symmetries. If K is any other hyperbolic knot, the natural question of whether there exists hidden symmetries is equivalent to asking whether Comm(K) contains Norm(K) as a proper subgroup. We reproduce some of the discussion in [42]. To do this requires some additional terminology.

A Euclidean torus T is isomorphic to $\mathbb{C}/\Lambda$ for some lattice Λ. The *conformal parameter* of T is the ratio of the two generators of Λ. This depends on a choice of generators but different choices only change this number by an element of PGL(2,$\mathbb{Z}$). As we have seen in Sections 3 and 5, the hyperbolic structure on the complement of a hyperbolic knot K induces a Euclidean structure on ∂X_K. If τ_K denotes the conformal parameter of ∂X_K then the discussion above shows that the *cusp field* of K, which is defined to be $\mathbb{Q}(\tau_K)$, is well-defined and is an invariant of K. It is not hard to check using the holonomy representation that $\mathbb{Q}(\tau_K)$ is a subfield of $\mathbb{Q}(\mathrm{tr}\Gamma_K)$.

The question about hidden symmetries can be re-formulated as follows (see [42] for details). If Norm(K) is a proper subgroup of Comm(K) then it follows that $S^3\backslash K$ must be an irregular cover of $\mathbb{H}^3$/Comm(K), which must be an orbifold with an end of a special type. We will not pursue this further, but using this forces some constraint on the cusp field. In particular, the following theorem can be proved,

Theorem 10.6. If K has hidden symmetries, then the cusp field of K is $\mathbb{Q}(\sqrt{-3})$ *or* $\mathbb{Q}(\sqrt{-1})$. □

The only knots known which have cusp field $\mathbb{Q}(\sqrt{-3})$ or $\mathbb{Q}(\sqrt{-1})$ are the figure-eight knot (from Example 2 of Section 5) and the two dodecahedral knots in Fig. 16 which also have cusp field $\mathbb{Q}(\sqrt{-3})$. In these latter two examples, the cusp field is a proper subfield of the trace-field (which is $\mathbb{Q}(\sqrt{-3},\sqrt{5})$) and give examples where the Comm(K) is strictly bigger than Norm(K). Based on this a natural conjecture is:

Conjectures. Let K be a hyperbolic knot distinct from the figure-eight knot or either of the two dodecahedral knots, then

1. *Comm(K) = Norm(K),*
2. *The cusp field of K coincides with trace-field of K, and is different from* $\mathbb{Q}(\sqrt{-3})$ *or* $\mathbb{Q}(\sqrt{-1})$.

As we discussed above, the 5_2 knot and the $(-2,3,7)$-pretzel knot have complements of the same volume and trace-field (which is cubic). However these knots are not commensurable. To see this, from Section 8, the group of symmetries of these knots are $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_2$ respectively, with all symmetries being orientation-preserving. By Theorem 10.5, these knots are not arithmetic and so as the trace-field has degree 3, Theorem 10.6 implies these knots cannot have hidden symmetries. Thus the groups Norm(K) (for K either the 5_2 knot or the $(-2,3,7)$-pretzel knot) must be the maximal groups. This together with the fact that their volumes are the same means that the knot complements cannot be commensurable.

It can be checked using the list of trace-fields of knots through 8 crossings listed in Appendix B that the cusp field of K coincides with the trace-field of K, and is different from $\mathbb{Q}(\sqrt{-3})$ or $\mathbb{Q}(\sqrt{-1})$ if K is not the figure-eight knot. In particular, by Theorem 10.6 Comm(K) = Norm(K) for all the knots in the table of Appendix B, so they do not admit any hidden symmetries.

Theorem 10.5 implies that the only knot with trace-field $\mathbb{Q}(\sqrt{-d})$ and whose traces are elements of O_d is the figure-eight knot. However it is conceivable, albeit unlikely, that there is a hyperbolic knot in S^3 with trace-field $\mathbb{Q}(\sqrt{-d})$, but its traces need not be algebraic integers in that field. We remark that there are examples of links in S^3 with this property (see [42]).

In fact until recently, all the evidence suggested that a hyperbolic knot has 'integral trace', that is the group Γ_K had algebraic integer traces. An interesting consequence of a positive solution to 'every knot has integral trace' would have been a solution to the Smith Conjecture for hyperbolic knots using only hyperbolic geometry (see Shalen's article in [38]), recall the discussion in Section 6. However, Craig Hodgson, using an exact version of SnapPea, called *Snap* (written by Oliver Goodman) has observed that the knot 10_{99} in the table of [57] does not seem to have 'integral trace'.

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APPENDIX A

Conway

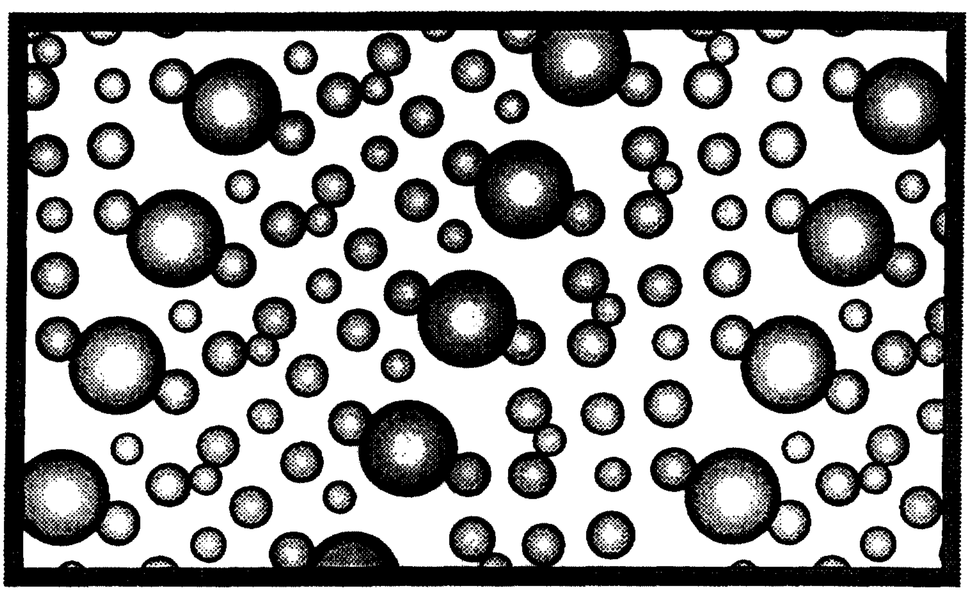

Fig. A1

Kinoshita-Teresaka

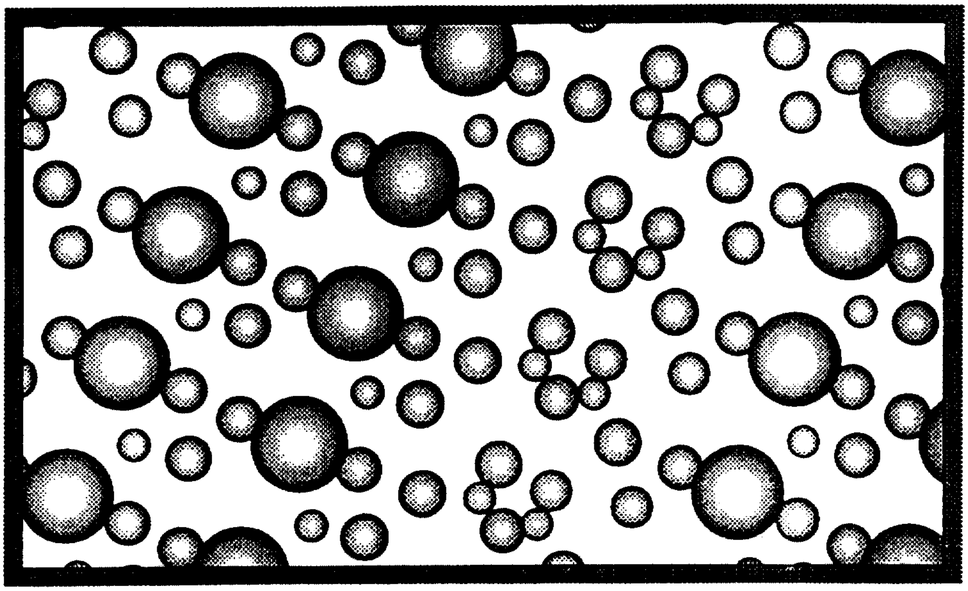

Fig. A2

APPENDIX B

Here we gather some information on hyperbolic knots through 8 crossings. In Table B1 below we collate some data for the knots through 8 crossings from [57]. The nomenclature for the knot is that of [57], and we give the volume if hyperbolic together with the symmetry group and details of the number of ideal tetrahedra in a decomposition into ideal tetrahedra. We give the minimal such number and the number in the canonical decomposition (recall Section 7). There is some overlap here with [9]. D_n denotes the dihedral group of order $2n$.

Here we give details on trace-fields of hyperbolic knot complements. We thank Craig Hodgson for providing this data. The nomenclature is as above. We list the trace-fields as $\mathbb{Q}(x)$ by stating the irreducible polynomial of x, and list an approximation to the particular root that is a primitive element. We also include the discriminant of the field.

4_1

minimum polynomial: $x^2 - x + 1$
numerical value of root: $0.5000000000000000 + 0.8660254037844386*i$
discriminant: -3.

5_2

minimum polynomial: $x^3 - x^2 + 1$
numerical value of root: $0.8774388331233463 - 0.7448617666197442*i$
discriminant: -23

6_1

minimum polynomial: $x^4 + x^2 - x + 1$
numerical value of root: $0.5474237945860586 - 0.5856519796895726*i$
discriminant: 257

6_2

minimum polynomial: $x^5 - x^4 + x^3 - 2x^2 + x - 1$
numerical value of root: $0.2765110734872844 - 0.7282366088878579*i$
discriminant: 1777

6_3

minimum polynomial: $x^6 - x^5 - x^4 + 2x^3 - x + 1$
numerical value of root: $1.073949517852393 + 0.5587518814119368*i$
discriminant: -10571

Table B1

Knot	Volume	Symmetry group	Number of tetrahedra: minimal/canonical
3_1	non-hyperbolic	–	–
4_1	2.029883212819307	D_4	2/2
5_1	non-hyperbolic	–	–
5_2	2.828122088330783	D_2	3/4
6_1	3.163963228883144	D_2	4/6
6_2	4.400832516123046	D_2	5/6
6_3	5.693021091281301	D_4	6/6
7_1	non-hyperbolic	–	–
7_2	3.331744231641115	D_2	4/8
7_3	4.592125697027063	D_2	5/8
7_4	5.137941201873418	D_4	6/8
7_5	6.443537380850573	D_2	7/8
7_6	7.084925953510830	D_2	8/8
7_7	7.643375172359955	D_4	8/8
8_1	3.427205246274016	D_2	5/10
8_2	4.935242678280654	D_2	6/10
8_3	5.238684100798440	D_4	6/10
8_4	5.500486416347235	D_2	6/10
8_5	6.997189147792215	D_2	8/12
8_6	7.475237429505243	D_2	8/10
8_7	7.022196589095253	D_2	8/10
8_8	7.801341224440063	D_2	9/10
8_9	7.588180223641627	D_4	8/10
8_{10}	8.651148558017082	$\mathbb{Z}_2$	9/11
8_{11}	8.286316817806593	D_2	9/10
8_{12}	8.935856927486689	D_4	10/10
8_{13}	8.531232201460416	D_2	9/10
8_{14}	9.217800316021929	D_2	10/10
8_{15}	9.930648293796183	D_2	11/12
8_{16}	10.579021916899270	$\mathbb{Z}_2$	11/14
8_{17}	10.985907606284820	$\mathbb{Z}_2$	12/16
8_{18}	12.350906209158200	D_8	13/14
8_{19}	non-hyperbolic	–	–
8_{20}	4.124903251807676	$\mathbb{Z}_2$	5/5
8_{21}	6.783713519835127	D_2	7/9

7_2

minimum polynomial: $x^5 - x^4 + x^2 + x - 1$
numerical value of root: 0.9355375391547716 + 0.9039076887509032*i
discriminant: 4409

7_3

minimum polynomial: $x^6 - x^5 + 3x^4 - 2x^3 + 2x^2 - x - 1$
numerical value of root: 0.4088024801541706 + 1.276376960703353*i
discriminant: 78301

7_4

minimum polynomial: $x^3 + 2x - 1$
numerical value of root: − 0.2266988257582018 + 1.467711508710224*i
discriminant: − 59

7_5

minimum polynomial: $x^8 - x^7 - x^6 + 2x^5 + x^4 - 2x^3 + 2x - 1$
numerical value of root: 1.031807435034724 + 0.6554697415289981*i
discriminant: − 4690927

7_6

minimum polynomial: $x^9 - x^8 + 2x^7 - x^6 + 3x^5 - x^4 + 2x^3 + x + 1$
numerical value of root: 0.7289655571286424 + 0.9862947000577544*i
discriminant: 90320393

7_7

minimum polynomial: x^4+x^2-x+1
numerical value of root: $-0.5474237945860586-1.120873489937059*i$
discriminant: 257

8_1

minimum polynomial: $x^6-x^5+x^4+2x^2-x+1$
numerical value of root: $0.9327887872637926+0.9516106941544145*i$
discriminant: -92051

8_2

minimum polynomial: $x^8-x^7+3x^6-2x^5+3x^4-2x^3-1$
numerical value of root: $0.4735144841426650-1.273022302875877*i$
discriminant: -21309911

8_3

minimum polynomial: $x^8-x^7+5x^6-4x^5+7x^4-4x^3+2x^2+1$
numerical value of root: $0.1997987161331217+1.513664037530055*i$
discriminant: 60020897

8_4

minimum polynomial: $x^9-x^8-4x^7+3x^6+5x^5-x^4-2x^3-2x^2+x-1$
numerical value of root: $1.491282033723026-0.2342960256675659*i$
discriminant: 1160970913

8_5

minimum polynomial: $x^5-x^4+2x^3+x^2+2$
numerical value of root: $0.1955670593924672+1.002696950053226*i$
discriminant: 8968

8_6

minimum polynomial: $x^{11}-x^{10}+2x^9-x^8+4x^7-2x^6+4x^5-x^4+3x^3+x^2+1$
numerical value of root: $0.7832729376220480-0.9737056666570652*i$
discriminant: -303291012439

8_7

minimum polynomial: $x^{11}-x^{10}-2x^9+3x^8+2x^7-4x^6+3x^4-x^3-x^2+1$
numerical value of root: $1.081079628832155-0.6317086402157812*i$
discriminant: -121118604943

8_8

minimum polynomial: $x^{12}-x^{11}-x^{10}+2x^9+3x^8-4x^7-2x^6+4x^5+2x^4-3x^3-x^2+1$
numerical value of root: $0.9628867449383822-0.8288503082039515*i$
discriminant: 2885199252305

8_9

minimum polynomial: $x^{12}-x^{11}-4x^{10}+x^9+10x^8+2x^7-12x^6-6x^5+7x^4+4x^3-2x^2+1$
numerical value of root: $-0.8475379649643470+0.8120675343521135*i$
discriminant: 421901335721

8_{10}

minimum polynomial: $x^{11}-2x^{10}+4x^8-2x^7-4x^6+5x^5+2x^4-5x^3+x^2+3x-1$
numerical value of root: $1.126054788892813+0.7113551926043732*i$
discriminant: -170828814392

8_{11}

minimum polynomial: $x^{10}-2x^9+3x^8-4x^7+4x^6-5x^5+5x^4-3x^3+3x^2-x+1$
numerical value of root: $0.3219944529118927+0.7144205683007117*i$
discriminant: -2334727687

8_{12}

minimum polynomial: $x^{14}-2x^{13}+3x^{12}-4x^{11}+4x^{10}-5x^9+7x^8-7x^7+7x^6-5x^5+4x^4-4x^3+3x^2-2x+1$
numerical value of root: $0.3846305385170291+0.9230706088052528*i$
discriminant: -15441795725579

8_{13}

minimum polynomial: $x^{14} - x^{13} - 3x^{12} + 4x^{11} + 4x^{10} - 7x^9 - x^8 + 6x^7 - 2x^6 - 2x^5 + 2x^4 - x + 1$
numerical value of root: 1.142594143553751 + 0.5467624949107860*i
discriminant: − 759929100364387

8_{14}

minimum polynomial: $x^{15} - x^{14} + 4x^{13} - 3x^{12} + 8x^{11} - 6x^{10} + 10x^9 - 7x^8 + 8x^7 - 6x^6 + 6x^5 - 4x^4 + 4x^3 - 2x^2 + 2x - 1$
numerical value of root: 0.5940318154659677 + 1.095616780826736*i
discriminant: − 26196407237223439

8_{15}

minimum polynomial: $x^7 - x^6 - x^5 + 2x^4 + x^3 - 2x^2 + x + 1$
numerical value of root: 1.139457724988333 + 0.6301696873026072*i
discriminant: − 1172888

8_{16}

minimum polynomial: $x^5 - 2x^4 + 2x^2 - x + 1$
numerical value of root: 1.417548120931355 − 0.4933740092574883*i
discriminant: 5501

8_{17}

minimum polynomial:
$x^{18} - 4x^{17} + 7x^{16} - 4x^{15} - 2x^{14} + x^{13} + 6x^{12} - 5x^{11} + 5x^{10} - 21x^9 + 36x^8 - 30x^7 + 22x^6 - 23x^5 + 18x^4 - 7x^3 + 2x^2 - 2x + 1$
numerical value of root: 0.98923482976437496 + 1.00826028978435916*i
discriminant: − 25277271113745568723

8_{18}

minimum polynomial: $x^4 - 2x^3 + x^2 - 2x + 1$
numerical value of root: − 0.2071067811865475 + 0.9783183434785159*i
discriminant: − 448

8_{20}

minimum polynomial: $x^5 - x^4 + x^3 + 2x^2 - 2x + 1$
numerical value of root: 0.4425377456177971 − 0.4544788918731118*i
discriminant: 5864

8_{21}

minimum polynomial: $x^4 - x^3 + x + 1$
numerical value of root: 1.066120941155950 + 0.8640541908597383*i
discriminant: 392

We remark that the fields for the knots 6_1 and 7_7 are conjugates of each other, however the roots given generate different subfields of the complex numbers.