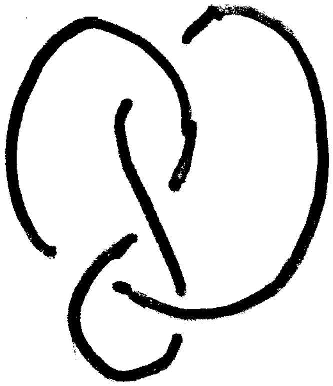
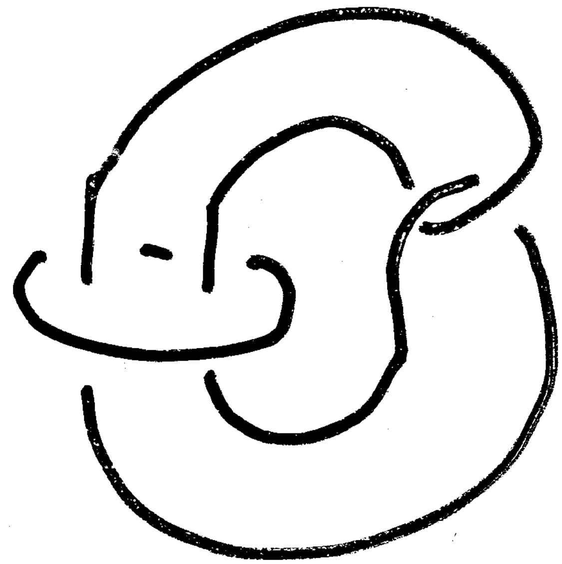
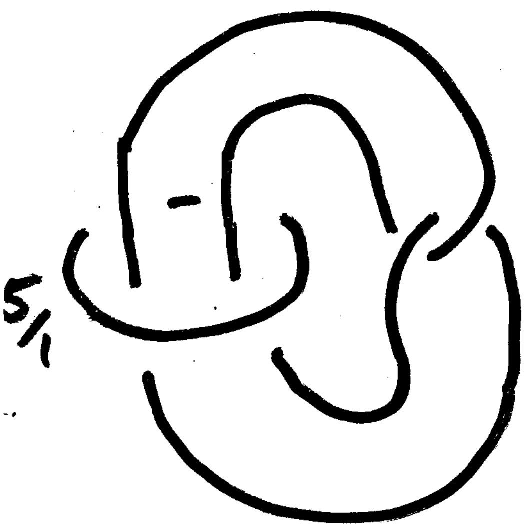
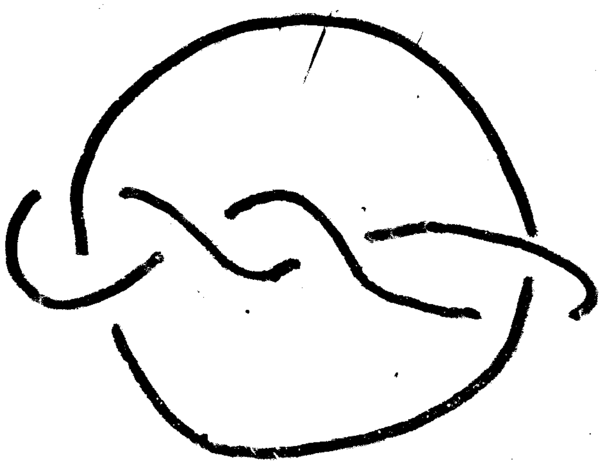
ARITHMETIC KNOTS AND LINKS

Alan W. Reid









$\boxed{\text{The Bianchi groups}}$

Let d be a square-free positive integer, and O_d the ring of algebraic integers in the field $\mathbf{Q}(\sqrt{-d})$. The collection of groups

$$PSL_2(\mathbb{Z}) < \boxed{\mathrm{PSL}(2, \mathrm{O_d})} \qquad \text{discrete subgps of } PSL_2(\mathbb{C}).$$

is called the *Bianchi groups.*

The quotients (*Bianchi orbifolds*):

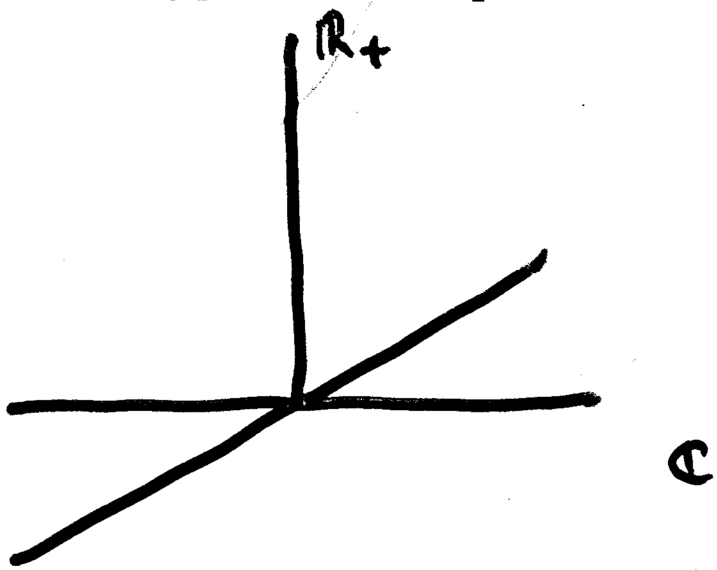
$$\boxed{Q_d = \mathbf{H}^3/\mathrm{PSL}(2, \mathrm{O_d})}$$

are finite volume hyperbolic orbifolds.

Definition: Let $M = \mathbf{H}^3/\Gamma$ be a non-compact finite volume hyperbolic 3-manifold (or orbifold). Then Γ is *arithmetic* if some conjugate of Γ in $\mathrm{PSL}(2, \mathbf{C})$ is commensurable with $\mathrm{PSL}(2, \mathrm{O_d})$.

Hyperbolic Manifolds

Let $\mathbf{H}^3$ denote hyperbolic 3-space.



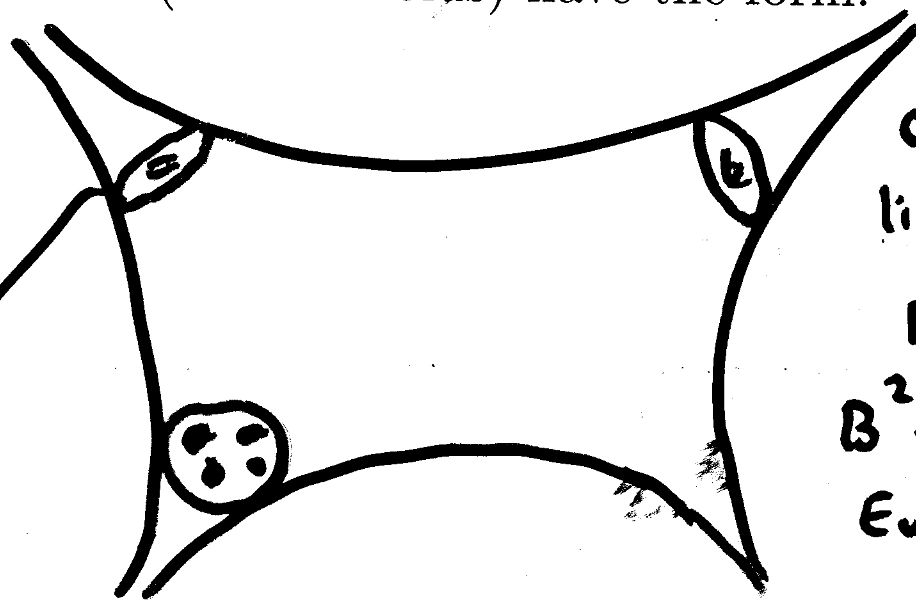
The full group of orientation-preserving isometries can be identified with $\mathrm{PSL}(2, \mathrm{C})$.

linear fractional action

$$z \longmapsto \frac{az+b}{cz+d}$$

extended to $\mathbb{H}^3$ by Poincare Extn.

We will only be interested in non-compact finite volume hyperbolic 3-manifolds(and orbifolds). These manifolds (and orbifolds) have the form:



Cusp End looks like

$B^2 \times [0,\infty)$

$B^2 = 2$ dim'l Euclidean orb.

These manifolds can be described as complements of links in closed orientable 3-manifolds.

$$\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \rangle \subseteq PSL_2(\mathbb{C})$$

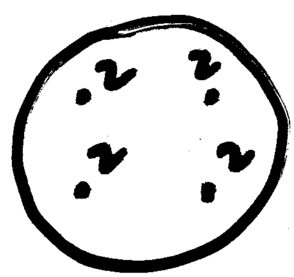
Ex $d = 1$ $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

Remarks:(1) Let h_d denote the class number of $\mathbf{Q}(\sqrt{-d})$. Hurwitz showed:

$$\boxed{Q_d \text{ has } h_d \text{ cusps.}}$$

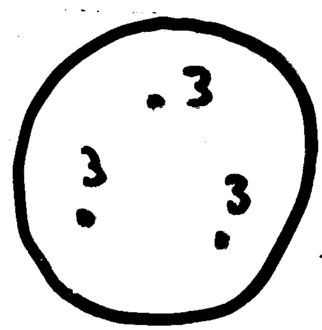
(2) When $d \neq 1, 3$, every cusp cross-section of Q_d is a torus.

When $d = 1$ the cusp cross-section is



$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ fixes ∞

When $d = 3$ the cusp cross-section is



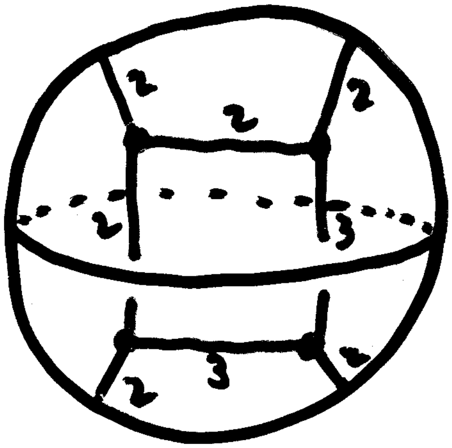
$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$ fixes ∞

$\omega = \frac{-1+\sqrt{-3}}{2}$

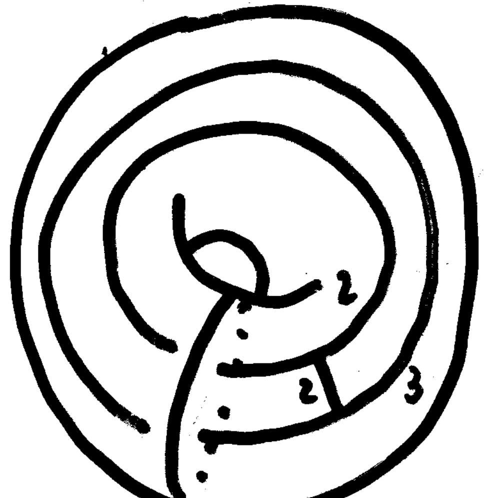
mark: When $h_d > 1$ there can be orbifolds in the commensurability class with 1 cusp.

There are only finitely many such comm classes.

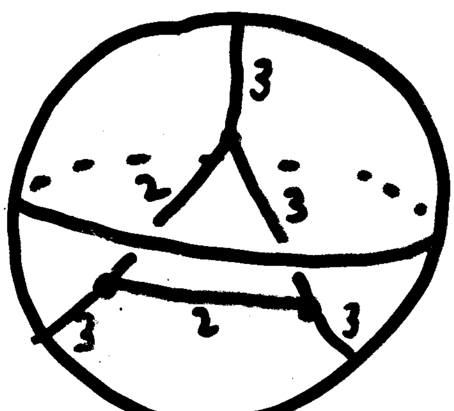
Some Bianchi orbifolds



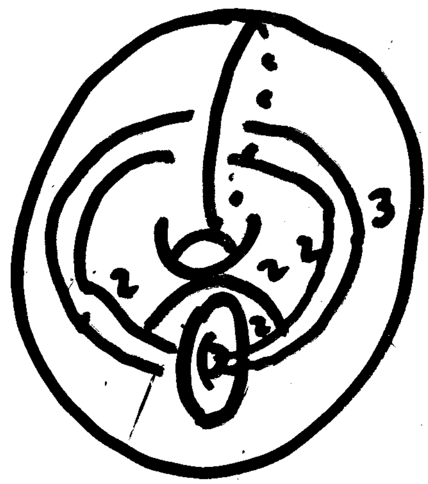
d = 1



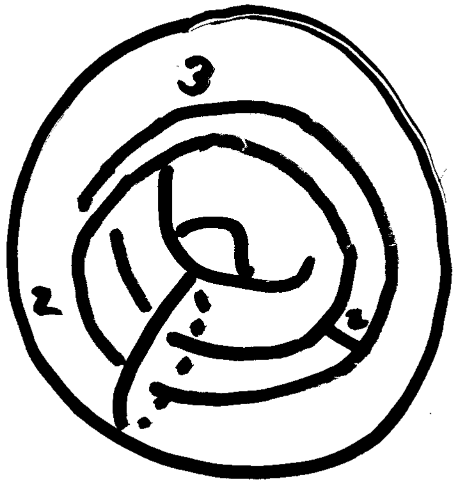
d = 2



d = 3



d = 5



d = 7

Comparison with the PSL(2, Z)

Cuspidal Cohomology

Let Γ be a non-cocompact Kleinian (resp. Fuchsian) group acting on $\mathbf{H}^3$ (resp. $\mathbf{H}^2$).

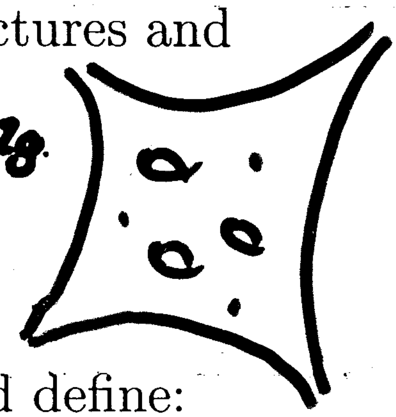
Let $\mathcal{U}(\Gamma)$ denote the subgroup of Γ generated by parabolic elements of Γ and define:

$$\boxed{V(\Gamma) = (\Gamma/\mathcal{U}(\Gamma))^{ab} \otimes_{\mathbf{Z}} \mathbf{Q}}$$

Then $\boxed{r(\Gamma) = \dim_{\mathbf{Q}}(V(\Gamma))}$ denotes the dimension of the space of non-peripheral homology or equivalently $r(\Gamma)$ is the dimension of the **Cuspidal Cohomology of Γ.**

Examples:

(1) If $\mathbf{H}^2/\Gamma$ is genus g surface with p punctures and finitely many orbifold points, then $r(\Gamma) = 2g$.



Thus $r(\mathrm{PSL}(2,\mathbf{Z})) = 0$.

(2) Let n be a square-free positive integer and define:

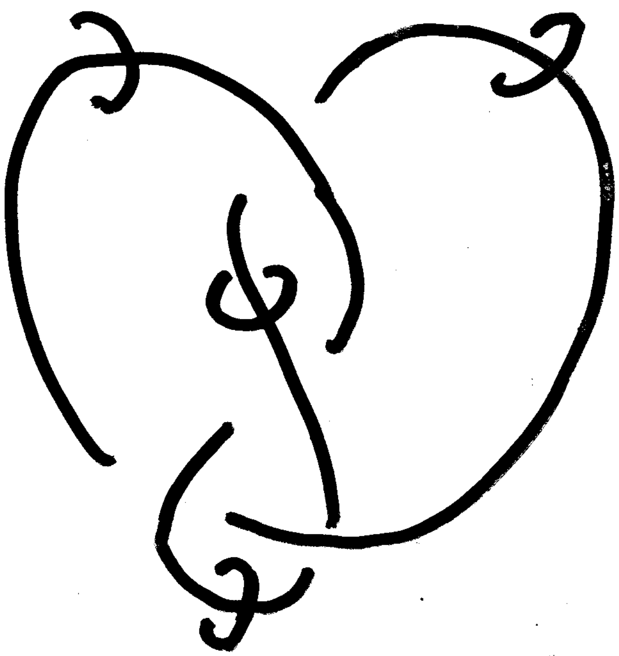
$$\Gamma_0(n) = \mathrm{P}\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;:\; \mathrm{c} = 0 \bmod \mathrm{n}\}$$

and $O_n = \mathbf{H}^2/\Gamma_0(n)$.

Now the Riemann-Hurwitz formula shows easily that $g(O_n) = 0$ if and only if $1 \leq n \leq 10$ and $n = 12, 13, 16, 18, 25$.

(2) If $L \subset S^3$ is a link then $r(\Gamma) = 0$. The link group is generated by meridians.

$= \pi_1(S^3, L)$



(3) If L is a link in a rational homology 3-sphere, $r(\Gamma) = 0$.

Grunewald's work on Bianchi groups with Schwermer:

Subgroups of Bianchi groups and arithmetic quotients of hyperbolic 3*-space.* Trans. Amer. Math. Soc. 335 (1993).

A nonvanishing theorem for the cuspidal cohomology of SL_2 *over imaginary quadratic integers.* Math. Ann. 258 (1981/82).

Arithmetic quotients of hyperbolic 3*-space, cusp forms and link complements.* Duke Math. J. 48 (1981).

Free nonabelian quotients of SL_2 *over orders of imaginary quadratic numberfields.* J. Algebra 69 (1981).

with Elstrodt and Mennicke:

Eisenstein series for imaginary quadratic number fields. Contemp. Math., 53.

Eisenstein series on three-dimensional hyperbolic space and imaginary quadratic number fields. J. Reine Angew. Math. 360 (1985).

On the group $\mathrm{PSL}_2(\mathbf{Z}[i])$. London Math. Soc. Lecture Note Ser., 56.

PSL(2) *over imaginary quadratic integers.* Asterisque, 94.

with Mennicke:

Some 3-manifolds arising from $\mathrm{PSL}_2(\mathbf{Z}[i])$. Arch. Math. (Basel) 35 (1980).

with Helling and Mennicke:

SL_2 *over complex quadratic number fields. I.* Algebra i Logika 17 (1978).

with Hirsch: *Link complements arising from arithmetic group actions.* Internat. J. Math. 6 (1995).

with Jaikin-Zapirain and Zalesskii: *Cohomological goodness and the profinite completion of Bianchi groups.* Duke Math. J. 144 (2008).

Preprint with Finis and Tirao

The cohomology of lattices in $SL_2(\mathbb{C})$.

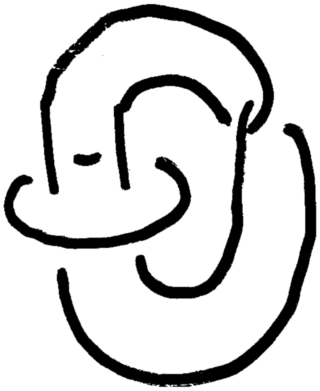
Theorem:(Cuspidal Cohomology Problem) (Harder, Zimmert, Grunewald-Schwermer, Rohlfs, ...,Vogtmann)
The only Bianchi orbifolds Q_d that can admit a finite sheeted cover that is a link complement in S^3 arise when

$$d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}.$$

(i.e. $c(PSL_2(O_d)) = 0 \Leftrightarrow d \in$ list above)

Theorem:(Baker)*For all values d as above, there exists a link complement covering Q_d.*

EXAMPLES

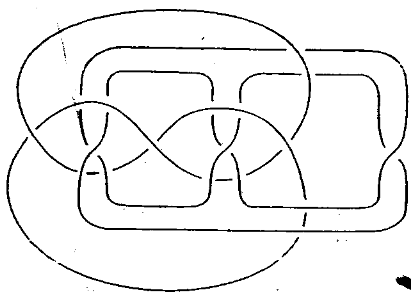


d = 1

d = 2

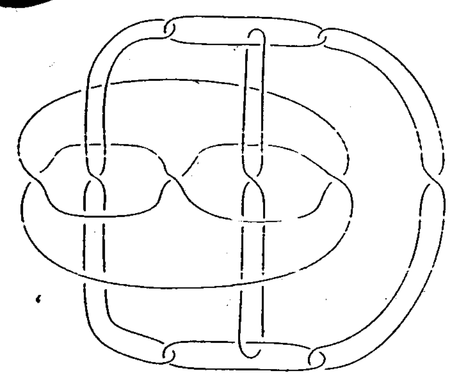
d = 3

d = 7

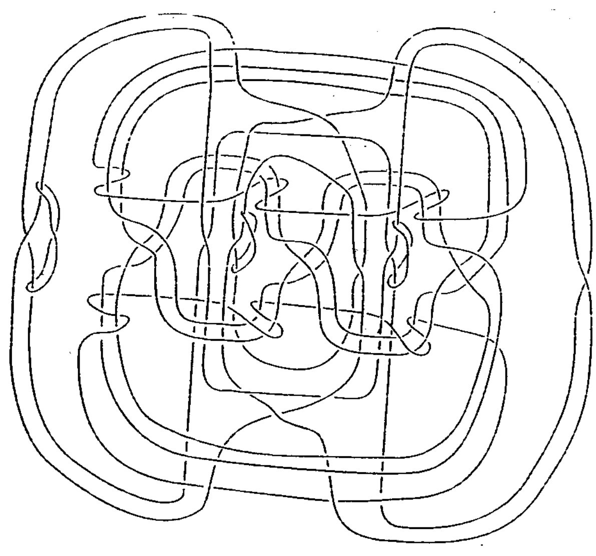


d = 15

Baker



d = 23



d = 47

Arithmetic Knots

What can one say about $M \to Q_d$ with M having 1 cusp. (or more generally 1-cusped arith. 3-mflds)

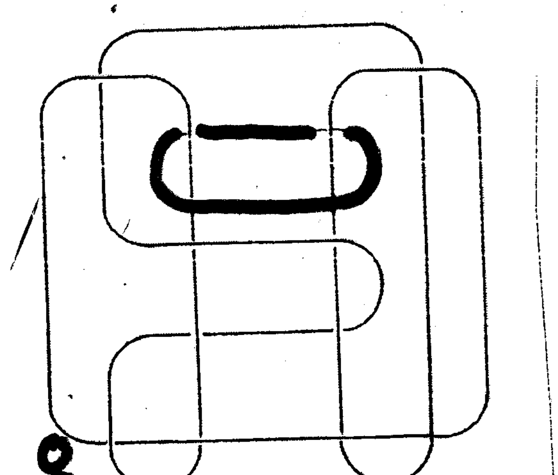
Note: If $M \to Q_d$ and M has 1 cusp, then Q_d has 1 cusp. Thus $h_d = 1$ (by Hurwitz's theorem).

Solution to the class number 1 problem: $h_d = 1$ if and only if

$$\boxed{d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}.}$$

Examples:

$d = 1$



$\frac{0}{1}$

10153 (Brunner, Frame, Lee Wielenberg)

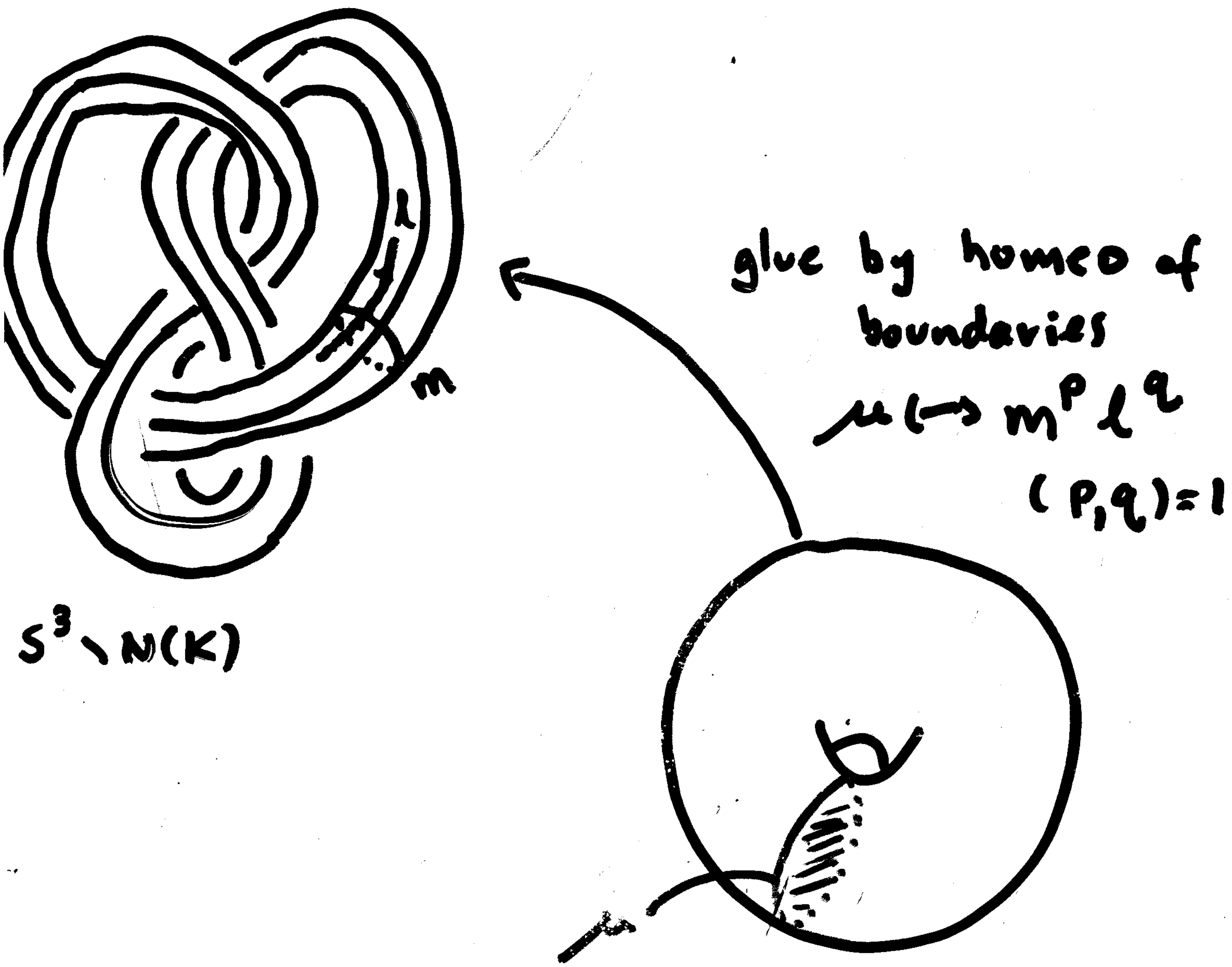
$d = 3$ $\frac{5}{1}$

[cyclic covers].

$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \right\rangle$

__Remark__: $d = 2, 7, 11, 19$ there are also 1 cusped covers.

Dehn Surgery



$S^3 \setminus N(K)$

V solid torus

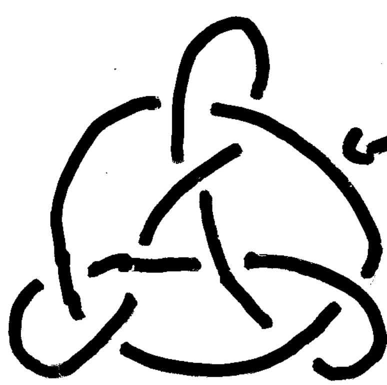
Understanding 1-cusped [redacted] arithmetic 3-manifolds is the same as understanding **arithmetic knots**.

Definiton: A knot K (or link L) in a closed orientable 3-manifold is called *arithmetic* if $M \setminus K$ (resp. $M \setminus L$) is arithmetic.

Example–S^3

Theorem:(R) *The figure eight knot is the only arithmetic knot.*

Links are different. *There are infinitely many arithmetic links (even of two components).*



Take m-fold cyclic covers $(3,m)=1$.

$d=2$.

A question that naturally arise from this:

Question: *Does every closed orientable 3-manifold contain an arithmetic knot?*

Why Care?

A positive answer implies the Poincare Conjecture.

The proof that the figure eight knot is the only arithmetic knot in S^3 shows that the figure eight knot in S^3 is the only arithmetic knot in a **homotopy 3-sphere**.

Remark: Once again links are different.

Every closed orientable 3-manifold contains an arithmetic link.

The reason is:

The figure eight knot is *universal* (every closed orientable 3-manifold arises as a branched cover of S^3 with branch set K).

Theorem 1:(Baker-R) *Suppose L is a Lens space with $\pi_1(L)$ of odd order $\neq 5$. Then L does not contain an arithmetic knot.*

Some ideas in Proof: L = Lens Space

Assume $|\pi_1 L|$ "large" > 37 prime

① $L \setminus K \longrightarrow Q_d$

"$\mathbb{H}^3/\Gamma$"

$h_d = 1$ + Trivial Cusp Cohom.

$\Rightarrow d \in \{1, 2, 3, 7, 11, 19\}$

② $P_\infty \subset \Gamma \subset PSL_2(O_d)$ be peripheral subgp fixing ∞.

ie $P_\infty = \left\langle \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right\rangle$, $x, y \in O_d$

$\cong \mathbb{Z} \oplus \mathbb{Z}$

μ = "meridian" of K.

Gromov - Thurston 2π - Thm $\Rightarrow |x| \leq 6$ "small"

(Improvement, Agol, Lackenby 6 Theorem)

$O_d \subset \mathbb{C}$ discrete $\Rightarrow$ only finitely many x.

Two Cases $\begin{cases} x \text{ a unit} \\ x \neq \text{a unit} \end{cases}$

(i) x a unit $\Rightarrow L \setminus K \cong S^3 \setminus 8$

Impossible as $S^3 \setminus 8$ has no Lens Space Dehn Surgery.

(ii) $x \neq$ a unit.

$\langle x \rangle$ non-trivial ideal, so $\exists\ \mathfrak{p} \mid \langle x \rangle$

$\mathfrak{p}$ a prime ideal.

$$\begin{tikzcd}
PSL_2(O_d) \arrow[rr, "\varphi_p"] & & PSL_2(O_d/\mathfrak{p}) \\
& & \varphi_p(\Gamma) \arrow[u, phantom, sloped, "\subseteq"] \\
\Gamma \arrow[uu, dash] \arrow[urr] \arrow[dr] \arrow[r, phantom, "\circlearrowright"] & & \\
& \Gamma/\langle\langle \mu \rangle\rangle = \pi_1 L \arrow[uur] &
\end{tikzcd}$$

$(\varphi_p(\mu) = 1$

$\Rightarrow \varphi_p(\Gamma)$ cyclic $(\neq 1 \quad \mathbb{H}^3/\mathrm{Ker}\,\varphi_p > 1$ cusp,

$\Rightarrow \varphi_p(\Gamma)$ is cyclic of large prime order

But $|\mathbb{F} = O_d/\langle \mathfrak{p} \rangle|$ is bounded as $|x| \leq 6$

Contradiction.

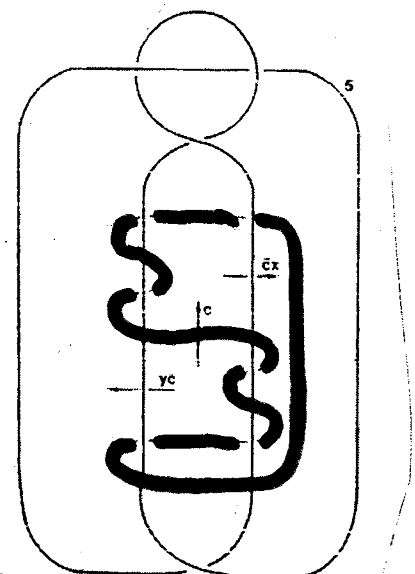
Unlike the case of S^3, there are examples of closed orientable hyperbolic 3-manifolds that contain more than one arithmetic knot.

Examples:

1. $S^2 \times S^1$ contains at least 2 arithmetic knots (the complements being commensurable with Q_3 and Q_7).
2. $\mathbf{RP}^3 \# \mathbf{RP}^3$ contains at least 2 arithmetic knots (the complements being commensurable with Q_1 and Q_3).
3. $L(4,1) \# L(4,1)$ contains at least 2 arithmetic knots (both the complements being commensurable with Q_7).
4. $\mathbf{RP}^3 \# (S^2 \times S^1)$ contains at least 2 arithmetic knots (the complements being commensurable with Q_1 and Q_3).

There are hyperbolic examples:

The manifold obtained by 5/1-Dehn surgery on the figure eight knot contains at least 2 arithmetic knots. One obvious one, and the other is shown:



Brunner - Frame - Lee
Wielenberg.

Question: *Is the number of arithmetic knots in a closed orientable 3-manifold finite?*

One can generalize the question about the uniqueness of the figure eight knot in S^3 in two obvious ways.

S^3 spherical

integral homology 3-sphere.

Question: *What can one say about arithmetic knots in spherical 3-manifolds or integral homology 3-spheres?*

Theorem 2:(Baker-R) *Suppose M be a spherical 3-manifold or an integral homology 3-sphere. Suppose that*

$$M \setminus K \to Q_d.$$

Then,

(1) If M is spherical then $d = 3$.

(2) If M is an integral homology 3-sphere, $d = 1, 3$.

Remark: If M is an integral homology 3-sphere and $K \subset M$ an arithmetic knot then one can show that $M \setminus K \to Q_d$ for some d.

[True for knots in mod 2 homology spheres]

Conjecture: *Let M be an integral homology 3-sphere. If M contains an arithmetic knot K, then M is obtained by $1/n$-Dehn surgery on the figure eight knot complement and K is "the core of the surgery solid torus".*

i.e. $\quad M \setminus K \cong S^3 \setminus$ fig 8 knot.

Final Comments

1-cusped congruence subgroups

In the case of the modular group H. Petersson showed that there are only finitely many 1-cusped congruence subgroups of $\mathrm{PSL}(2, \mathbf{Z})$.

In her (2005) thesis, K. Petersen (my former student) showed that there are only finitely many *maximal* 1-cusped congruence subgroups.

Indeed for $d = 11, 19, 43, 67, 163$ there are only finitely many 1-cusped congruence subgroups.

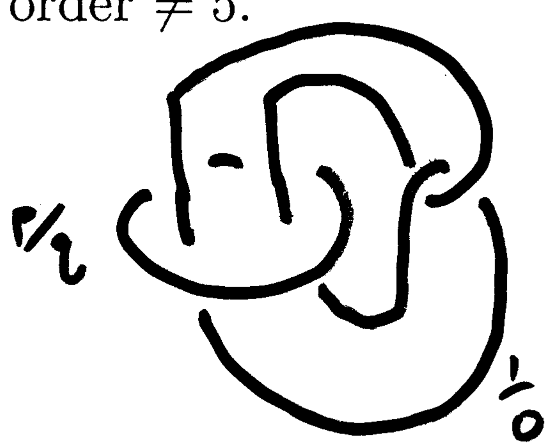
For $d = 19, 43, 67, 163$ there are no torsion-free 1-cusped congruence subgroups

Arithmetic number of a closed orientable 3-manifold

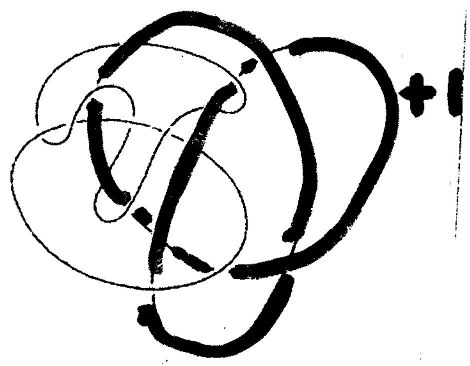
Let M be a closed orientable 3-manifold. The *the arithmetic number of* M, denoted $\mathcal{A}(M)$, is the minimal number of components of a non-empty arithmetic link in M.

As remarked, M contains an arithmetic link, so $\mathcal{A}(M)$ is well defined positive integer.

Examples:(1) A Lens Space L is a Dehn surgery on the Whitehead link, so that $\mathcal{A}(L) \leq 2$. Theorem 1 therefore shows $\mathcal{A}(L) = 2$ for L with $\pi_1(L)$ odd order $\neq 5$.



(2) The Poincare homology sphere Σ contains a 2-component arithmetic link so $\mathcal{A}(\Sigma) \leq 2$.



Brunner - Frame - Lee - Wielenberg.

This prompts:

Question: *Does the Poincare homology sphere contain an arithmetic knot?*

(3) Methods of proof of Theorem 1 show "many" non-hyperbolic 3-manifolds have arithmetic number ≥ 2.

Challenges:

(1) Prove that there exists closed orientable 3-manifolds for which $\mathcal{A}(M)$ is arbitrarily large.

(2) Provc that there exists a closed orientable hyperbolic 3-manifold that does not contain an arithmetic knot; ie $\mathcal{A}(M) \geq 2$.

(1) looks like it is related to Heegaard genus.

(2) There are candidate integral homology 3-spheres.