373K Algebra I, Homework 11

From Artin

Chapter 11 (pages 354–357) 3.2, 3.5, 3.8 (just take $R = \mathbf{F}_p$), 3.9, 3.12, 3.13, 7.5.

Others:

1. Let $R = \mathbf{F}_2[x]$ and $p(x) \in R$ given by $x^2 + x + 1$. Prove that $R/ < p(x) >$ is a field with four elements.

2. The *center* of a ring R (with 1) is:

$$Z(R) = \{z \in R : rz = zr \quad \forall r \in R\}.$$

(a) Prove that $Z(R)$ is a subring of R that contains the identity.

(b) Prove that the center of a divison ring is a field.

3. Let G be a finite group, and R a commutative ring with 1. Prove that $Z(RG)$ is non-trivial. (**Hint:** sum all the elements of G).

4. Which of the following are ideals of $\mathbf{Z}[x]$?

(a) the set of all polynomials with constant term a multiple of 3.

(b) the set of all polynomials whose coefficient of x^2 is a multiple of 3.

(c) the set of polynomials in which only even powers of x appear.

(d) the set of polynomials $p(x)$ such that $p'(0) = 0$ (where $p'(x)$ is the usual derivative function).

5. Prove that $M_2(\mathbf{R})$ contains a subring isomorphic to $\mathbf{C}$.

6. Let R be a commutative ring with 1.

(a) Define

$$\mathrm{rad\ I} = \{\mathrm{r} \in \mathrm{R} : \mathrm{r}^{\mathrm{n}} \in \mathrm{I} \text{ for some } \mathrm{n} \in \mathbf{N}\}.$$

Prove that rad I is an ideal of R containing I (called the *radical of I*.

(b) An element $a \in R$ is called *nilpotent* if $a^m = 0$ for some $m \geq 1$. Prove that the set of nilpotent elements form an ideal (*the nilradical*), and that $1 + a$ is a unit.

(c) What is rad I/I?

7. Determine the nilpotent elements of $\mathbf{Z}/72\mathbf{Z}$.