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# Splitting Groups of Signature $(1; n)$

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## 1. INTRODUCTION

Let $G$ be a co-compact Fuchsian group. As such, it acts discontinuously on the hyperbolic plane $\mathbf{H}^2$ and the quotient is a closed hyperbolic 2-orbifold. The structure theory for co-compact Fuchsian groups yields a canonical presentation for such groups. This information is described by the signature

$$\sigma = (g; m_1, \dots, m_r) \tag{1}$$

where $g$ is the genus of the base surface of the quotient 2-orbifold, and the conjugacy classes of the maximal non-trivial finite cyclic subgroups have orders $m_1, \dots, m_r$. In the case when $g$ is zero we will abbreviate the signature to $(m_1, m_2, \dots, m_r)$.

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The study of decompositions of co-compact Fuchsian groups as free products with amalgamation can be traced back to work by Fenchel and Nielsen [6], and more recently to the work of Zieschang ([23, 24]) and Rosenberger [15]). Now for most co-compact Fuchsian groups, it is easy to decompose the group as a free product with amalgamation. In these cases, this is achieved geometrically by finding a separating simple closed curve. An exceptional case is that of the triangle groups, that is groups of signature $(p, q, r)$ for positive integers $p$, $q$, and $r$ satisfying $1/p + 1/q + 1/r < 1$, which are easily seen not to admit any splitting as a free product with amalgamation or HNN-extension (for instance, they have Property (FA) of [17]). However, there is a class of Fuchsian groups, namely those of signature $(1; n)$ for which there is no geometric splitting and so the question of whether they admit a free product with amalgamation decomposition is a non-trivial one. This is an old problem, which for example was explicitly stated in [25, Chap. 4.12] (and references cited above), and more recently restated as Question 26 of The Problem Session in [7]:

QUESTION 1.1. *Let $G$ be a Fuchsian group of signature $(1; n)$. Does $G$ have a decomposition as a free product with amalgamation*?

As is easy to see $G$ has a presentation on two generators;

$$\langle a, b \mid [a, b]^n = 1 \rangle.$$

Using this, it is known that there does exist such a decomposition for certain values of $n$. The following discussion is contained in [15], correcting [23].

Suppose $n$ is an odd positive integer greater than 1, and suppose $H$ is a Fuchsian group of signature $(2, 2, 2, n)$. It is immediate that $H$ admits a free product with amalgamation decomposition with an infinite cyclic group amalgamated. This group is known to be 2-generator, since if the canonical presentation is given by

$$\langle x_1, x_2, x_3, x_4 \mid x_1^2 = x_2^2 = x_3^2 = x_4^n = 1,\ x_1x_2x_3x_4 = 1 \rangle,$$

then a pair of generators is $\{x_1x_2, x_2x_3\}$. Furthermore, one can check that a group $G$ of signature $(1; n)$ (with $n$ as above) surjects $H$, and hence one can pull back the splitting of $H$. In addition, if $n$ is not a power of 2, there is an epimorphism from the group of signature $(1; n)$ to a group of signature $(1; \text{odd})$ from which the splitting derived above can be pulled back. Here we give a proof of the following theorem valid for all $n \geq 2$ which answers Question 1.1 and therefore can be seen as the natural conclusions of the sequence of works [6, 23, 24, and 15]:

THEOREM 1.2. *Let $G$ be a Fuchsian group of signature $(1; n)$, then $G$ admits a decomposition as a free product with amalgamation. Moreover, there exists such a decomposition where the factor groups and the amalgamating subgroup are all finitely generated.*

The proof of the theorem actually applies to a more general class of groups (see Theorem 5.1). The method of proof is to invoke the Bass–Serre theory [17, 1, and 2], applied to a suitable representation of $G$ in $\mathrm{PSL}_2(\mathbf{Q_p})$ where $\mathbf{Q}_p$ denotes the $p$-adic rationals. The Bass–Serre theory yields an action on a tree. To deduce we have a free product with amalgamation decomposition we apply further results of Serre [17] together with the theory of $p$-adic Lie groups. Although the constructed action on the tree may not have finitely generated vertex and edge stabilizers, we can invoke Theorem 1 of [3] which guarantees that if there is a splitting as a free product with amalgamation, there is one with finitely generated factor and amalgamating subgroups.

We have attempted to make the discussion of $p$-adic Lie groups reasonably self-contained for the benefit of the reader. Consequently, some material in Section 4 is a synopsis of known results.

## 2. VALUATIONS AND THE TREE OF $\mathrm{SL}_2$

In this section, for convenience, we recall material of [17] and [2] concerning the tree of $\mathrm{SL}_2$ over a local field. We will only be interested in non-archimedean local fields arising as completions of a number field. Therefore we describe Serre's constructions in the context of these fields. We also recall some basic facts from the theory of number fields and their completions (see for instance [11] for details).

Let $k$ be a number field, that is, a finite extension of $\mathbf{Q}$. Denote by $\mathscr{O}_k$ the ring of algebraic integers of $k$. For each prime $\mathscr{P}$ of $k$ we denote by $\nu_{\mathscr{P}}$ the discrete valuation (the *$\mathscr{P}$-adic valuation*) associated to $\mathscr{P}$, normalized to take values in $\mathbf{Z}$. Let $k_{\mathscr{P}}$ denote the completion of $k$ with respect to the valuation $\nu_{\mathscr{P}}$ and $\mathscr{O}_{\mathscr{P}}$ the ring of $\mathscr{P}$-adic integers, that is, the set $\{x \in k_{\mathscr{P}} : \nu_{\mathscr{P}}(x) \geq 0\}$. The ring $\mathscr{O}_{\mathscr{P}}$ has a unique maximal ideal generated by an element $\pi$, called a *local uniformizing parameter*, such that $\nu_{\mathscr{P}}(\pi) = 1$. An element $x$ of $\mathscr{O}_{\mathscr{P}}$ is called a *$\mathscr{P}$-adic unit* if $\nu_{\mathscr{P}}(x) = 0$. Any element $x \in k_{\mathscr{P}}$ can then be written as $u\pi^s$ for some $\mathscr{P}$-adic unit $u$ and $s \in \mathbf{Z}$. Let $P$ be the set of all prime ideals $\mathscr{P}$ of $\mathscr{O}_k$ and, for each $\mathscr{P} \in P$, fix an embedding of $k$ in $k_{\mathscr{P}}$.

The following lemma is well-known (cf. [1, Lemma 6.8.2]):

LEMMA 2.1. *Let $k$ be a number field and let $\alpha \in k$. Then $\alpha \in \mathscr{O}_k$ if and only if $\alpha \in \mathscr{O}_{\mathscr{P}}$ $\forall \mathscr{P} \in P$.*

We next recall the construction of the tree of $\mathrm{SL}_2$. Let $\mathscr{P} \in P$, $K = k_{\mathscr{P}}$, and $\mathscr{O} = \mathscr{O}_{\mathscr{P}}$. The group $\mathrm{SL}_2(K)$ acts on a tree T constructed as follows; see [17, Chap. 2]. Let $V$ denote the vector space $K^2$; a lattice in $V$ is a finitely generated $\mathscr{O}$-submodule which spans $V$. Define an equivalence relation on the set of lattices of $V$ as follows: $\Lambda \sim \Lambda'$ if and only if $\Lambda' = x\Lambda$ for some $x \in K^*$. Let $[\Lambda]$ denote the equivalence class of $\Lambda$. These equivalence classes form the vertices of a combinatorial graph T where two vertices $[\Lambda], [\Lambda']$ are connected by an edge if there is an $x \in K^*$ such that $x\Lambda' \subset \Lambda$ and $\Lambda / x\Lambda' \cong \mathscr{O}_{\mathscr{P}} / \pi \mathscr{O}_{\mathscr{P}}$. It is shown in [17] that T is a tree; i.e., it is connected and simply connected.

The obvious action of $\mathrm{GL}_2(K)$ on the set of lattices in $V$ determines an action on T which is transitive on vertices. Restricting the action to $\mathrm{SL}_2(K)$ yields an action of $\mathrm{SL}_2(K)$ on T with no inversions. The vertices form two orbits and the stabilizers are represented by $\mathrm{SL}_2(\mathscr{O})$ and a conjugate of $\mathrm{SL}_2(\mathscr{O})$, namely $\{\left(\begin{smallmatrix} a & \pi b \\ \pi^{-1} c & d \end{smallmatrix}\right) : ad - bc = 1, a, b, c, d \in \mathscr{O}\}$. In particular, we deduce:

Lemma 2.2. *Let $G$ be a subgroup of $\mathrm{SL}_2(K)$ which fixes a vertex, then the traces of elements of $G$ are elements of $\mathscr{O}$.*

The version of the splitting theorem we will work with is given below. First we recall Property (D) defined by Serre on p. 78 of [17].

Let $j : \mathrm{GL}_2(K) \to \mathbf{Z}_2$ be the composite epimorphism obtained by composing reduction modulo 2 with the map, $\nu_{\mathscr{P}} \circ \det$. The kernel of $j$ is denoted by $\mathrm{GL}_2^+(K)$. Note that this contains $\mathrm{SL}_2(K)$.

Definition 2.3. Let $G$ be a subgroup of $\mathrm{GL}_2^+(K)$. We say $G$ has Property (D) if the closure of $G$ in the $\mathscr{P}$-adic topology in $\mathrm{GL}_2(K)$ contains $\mathrm{SL}_2(K)$.

With this, Theorem 3 on p. 79 of [17] yields:

Theorem 2.4. *Let $G$ be a subgroup of $\mathrm{GL}_2^+(K)$. If $G$ has property* (D) *then $G$ splits as a non-trivial free product with amalgamation $G_\Lambda *_{G_{\Lambda\Lambda'}} G_{\Lambda'}$, where $G_*$ denotes the vertex stabilizer or edge stabilizer.*

We remark here that if $G$ satisfies Theorem 2.4 and the center $Z(G)$ is non-trivial, then since the center of an amalgamated product is contained in the amalgamating group it follows that $G/Z(G)$ also splits as a free product with amalgamation.

## 3. ALGEBRAIC REPRESENTATIONS

By definition a Fuchsian group $G$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$ and thus the elements of $G$ can be realized by elements in $\mathrm{SL}_2(\mathbf{R})$ up to a factor of $\pm 1$. Following the notation of [21] and [22], if $k$ is a number field, we say $\sigma$ has a *representative in* $k$, if there exists a Fuchsian group $G_0$ of signature $\sigma$ such that $G_0 \subset \mathrm{PSL}_2(k)$.

It is clear that any field $k$ for which $\sigma$, as at (1), has a representative, contains the field

$$K = \mathbf{Q}(\cos \pi/m_1, \cos \pi/m_2, \dots, \cos \pi/m_r).$$

Let $\mathscr{T}(G)$ be the Teichmüller space of $G$. As is well known the only Fuchsian groups for which $\mathscr{T}(G)$ is a point are triangle groups. In all that follows, it will be a standing assumption that $G$ is not a triangle group. In particular, $\mathscr{T}(G)$ has dimension greater than 0. Here we are viewing $\mathscr{T}(G)$ as a quotient variety of the representation variety of $G$ in $\mathrm{PSL}_2(\mathbf{R})$. Let $k$ be a field and denote by $\mathscr{T}(G)_k$ the subspace corresponding to groups whose elements lie in $\mathrm{PSL}_2(k)$.

DEFINITION 3.1. Let $k$ be a number field, and $G < \mathrm{PSL}_2(k)$. Say $G$ is nonintegral at $\mathscr{P}$, if there is an element $g \in G$ and a $k$-prime $\mathscr{P}$ such that $\nu_{\mathscr{P}}(\mathrm{tr}(g)) < 0$.

With this definition (in the notation above) we prove:

LEMMA 3.2. *With $\sigma$ as above, we can choose $\mathscr{P}$ such that $\sigma$ has a representative $G$ in $K$ which is non-integral at $\mathscr{P}$ and if $p = \mathscr{P} \cap \mathbf{Z}$, $p$ is odd and splits completely in $K$.*

Recall that if $[k : \mathbf{Q}] = n$ with ring of integers $\mathscr{O}_k$ and $p$ a rational prime, then $p$ is said to *split completely* in $k$ if $p\mathscr{O}_k = \mathscr{P}_1 \dots \mathscr{P}_n$ for distinct $\mathscr{P}_i$. It is well known, cf. [14, Theorem 4.12] that there are infinitely many rational primes that split completely in any number field.

The proof of Lemma 3.2 is essentially implicit in [21], and also in [19] for the case of $k = \mathbf{Q}$.

For convenience of notation we deal explicitly with the case of a Fuchsian group of signature $(1; n)$. The general case follows from a similar argument in the cases where the genus is positive, and a related argument when $g = 0$ [21, 19].

Thus let $G = \langle a, b, t \,|\, [a, b] = t,\ t^n = 1 \rangle$, and $\phi : G \to \mathrm{PSL}_2(\mathbf{R})$ be an isomorphism with discrete image $\Gamma$. Let $F = P^{-1}(\Gamma)$, where $P : \mathrm{SL}_2(\mathbf{R}) \to \mathrm{PSL}_2(\mathbf{R})$. Then $F$ is a central extension of $\langle -I \rangle$ by $\Gamma$. When $n$ is even this extension does not split and $F$ has the presentation

$$\langle A, B, T, J \,|\, [A, B] = TJ,\ T^n = J,\ J^2 = I,\ J \text{ commutes with } A, B \rangle. \quad (2)$$

By conjugating we can assume that $B$ is diagonal. Suppose also that $[A, B] = D = TJ$ with $A = [a_{ij}]$, $B = [b_{ij}]$, and $D = [d_{ij}]$. Discreteness implies that none of $a_{12}, a_{21}, d_{12}, d_{21}$ can be zero. The following is contained in [21] and [19]:

LEMMA 3.3. *Given* $Z = [z_{ij}] \in \mathrm{SL}_2(\mathbf{R})$ *with* $z_{12} \neq 0$, *and* $x, y \in \mathbf{R}$, $x \neq 0$, $y \neq 0, 1$, *there exists a unique solution to* $[X, Y] = Z$, *with* $X, Y \in \mathrm{SL}_2(\mathbf{R})$, $X = [x_{ij}]$, $Y = [y_{ij}]$, $x_{12} = x$, $y_{11} = y$, $y_{12} = 0$; *namely,*

$$Y = \begin{pmatrix} y & 0 \\ \dfrac{1 - z_{11}}{z_{12}} y + \dfrac{1 - z_{22}}{z_{12}} \dfrac{1}{y} & \dfrac{1}{y} \end{pmatrix}$$

$$X = \begin{pmatrix} \dfrac{y y_{12}}{1 - y^2} x + \dfrac{z_{12}}{1 - y^2} \dfrac{1}{x} & x \\ \dfrac{(z_{22} - y^2) y y_{21}}{z_{12}(1 - y^2)} x + \dfrac{z_{22} - 1}{1 - y^2} \dfrac{1}{x} & \dfrac{z_{22} - y^2}{z_{12}} x \end{pmatrix}.$$

*Proof of Lemma* 3.2. With reference to the normalized representation of $F$ above, with $B$ diagonal, we can perturb the matrix $T$ slightly so that the entries of $Z = TJ$ lie in $K$, the entries $z_{12}, z_{21} \neq 0$, and $T$ still defines an elliptic element of order $n$.

If we now specify $x$ and $y$, then by Lemma 3.3 there is a unique solution to $[X, Y] = Z$, where the entries of $X, Y$ are rational functions in $x, y$ and the entries of $Z$. Furthermore note that $\mathrm{tr}(Y) = y + y^{-1}$. Let $p$ be any large odd prime which splits completely in $K$—as remarked above, there are infinitely many such primes. Choose $x, y \in \mathbf{Q}$ close to $a_{12}, b_{11}$ and such that $y = v/p$ with $v$ a $p$-adic unit.

This yields a representation $G \hookrightarrow \mathrm{PSL}_2(K)$ which is close to the original. Since $\mathscr{T}(G)$ has dimension greater than zero, the subspace of isomorphic representations with discrete image is open in the representation space. It follows that the image group is discrete of signature $(1; n)$, and has an element whose trace is $a/p$ for some $a \in \mathbf{Q}$ which is a $p$-adic unit. This concludes the proof of Lemma 3.2. ∎

*Remark.* The reason for insisting on a trace with denominator as above will become clear in Section 5. The main point is that it makes for a more direct application of the technology of $p$-adic Lie groups discussed in Section 4.

## 4. $p$-ADIC LIE GROUPS AND LIE ALGEBRAS

The proof of our main result is the next section requires some results on $p$-adic Lie groups and algebras. Here we gather together the background information required. For general information on Lie algebras and Lie groups, see [9, 10, and 18]. For the particular discussion on $p$-adic Lie groups and algebras that follows, we refer to [5].

Throughout $p$ will denote an odd prime.

DEFINITION 4.1. Let $H$ be a topological group. Then $H$ is defined to be a $p$-adic Lie group if $H$ has the structure of an analytic manifold over $\mathbf{Q}_p$ and if the function $H \times H \to H$ defined by $(x, y) \to xy^{-1}$ is analytic.

By a *Lie algebra* over $\mathbf{Q}_p$ we mean a vector space over $\mathbf{Q}_p$ with a multiplication which satisfies the conditions

$$x^2 = 0 \qquad \text{and} \qquad (xy)z + (yz)x + (zx)y = 0.$$

*Notation.* If $G$ is a group, then $G^p = \langle g^p \mid g \in G \rangle$.

Recall that a *profinite group* is a compact Hausdorff topological group whose open subgroups form a base for the neighborhoods of the identity and can be characterized as an inverse limit of an inverse system $\{G_i\}$ of finite groups. If the finite groups $G_i$ are all $p$-groups, we obtain a *pro-p group* and if, furthermore, the maps in the inverse system are all surjective and the quotients $G_i/G_i^p$ abelian, then the inverse limit is a *powerful pro-p group*. Finally, a pro-$p$ group is termed *uniform* if it is finitely generated, powerful, and satisfies

$$[P_i(G) : P_{i+1}(G)] = [G : P_2(G)] \qquad \text{for all } i,$$

where $P_1(G) = G$ and $P_{i+1}(G)$ is defined recursively as $P_i(G)^p[P_i(G), G]$. One should take closures in the previous statement, but under the assumption that $G$ is finitely generated [5, Corollary 1.20] there is no need.

For such groups, a *dimension* can be defined as the minimum cardinality of a topological generating set. The following fundamental result characterizes $p$-adic Lie groups in terms of these uniform pro-$p$ groups, cf. [5, Theorem 9.34].

THEOREM 4.2. *Let $G$ be a topological group. Then $G$ is a $p$-adic Lie group if and only if $G$ contains an open subgroup which is a uniform pro-$p$ group.*

We remark that the *dimension* of $G$ is defined to be the dimension of the uniform pro-p subgroup in the above theorem. By [5, Theorem 9.38] this is also the dimension of every chart belonging to an atlas defining the manifold structure on $G$ (recall Definition 4.1).

For applications in the next section, we actually only need to consider properties of the group $\mathrm{SL}_2(\mathbf{Q}_p)$. However, the same techniques allow us to discuss $p$-adic groups of dimension 3 more generally, which we do for completeness.

Recall that there are, up to isomorphism, exactly two quaternion algebras over $\mathbf{Q}_p$ (cf. [20, Chap. 2]),

$$M = M_2(\mathbf{Q}_p) \qquad \text{and} \qquad D = \left(\frac{u, p}{\mathbf{Q}_p}\right), \tag{3}$$

where $u$ is a non-square unit in $\mathbf{Z}_p$. We will show that the elements of norm 1 in these algebras, $\mathrm{SL}_2(\mathbf{Q}_p)$ and $D^1$, are $p$-adic Lie groups of dimension 3.

In what follows $A$ will denote either $M$ or $D$. Let $\mathscr{O}$ be a maximal order in $A$. In the case of $M$, we choose $\mathscr{O} = M(2, \mathbf{Z}_p)$. In the second case, $D$ can be represented as

$$D = \left\{ \begin{pmatrix} a & b \\ p\bar{b} & \bar{a} \end{pmatrix} \middle| a, b \in L \right\},$$

where $L$ is the unique unramified quadratic extension of $\mathbf{Q}_p$ and $\bar{c}$ denotes the $L \mid \mathbf{Q}_p$ conjugate of $c \in L$ (see [20, Chap. 2]). The unique maximal order in this case consists of those matrices with $a, b \in R_L$, the ring of integers in $L$.

The principal congruence subgroups are defined in each case by:

DEFINITION 4.3. $\Gamma_i = (1 + p^i\mathscr{O}) \cap \mathscr{O}^1$.

Now $\Gamma_1$ is the inverse limit of the $p$-groups $\{\Gamma_1/\Gamma_i\}$ and each $\Gamma_i$ is open in $\Gamma_1$. Furthermore, for each $i$, it is straightforward to show that $\Gamma_i^p = \Gamma_{i+1}$ and that $\Gamma_i/\Gamma_{i+1}$ is an elementary abelian $p$-group of order $p^3$. In the case of $D$,

$$\Gamma_i/\Gamma_{i+1} \cong \left\{ \begin{pmatrix} 1 + p^i a & p^i b \\ 0 & 1 + p^i\bar{a} \end{pmatrix} \in \mathrm{SL}_2\left(R_L/p^{i+1}R_L\right) \right\}.$$

(cf. [5, pp. 85–86]). Thus $\Gamma_2 = \Phi(\Gamma_1)$, the Frattini subgroup of "non-generators." This discussion together with Propositions 1.13 and 1.14 of [5] then implies that $\Gamma_1$ is a finitely generated powerful pro-$p$ group for which $P_i(\Gamma_1) = \Gamma_i$. Hence $\Gamma_1$ is a uniform pro-$p$ subgroup of dimension 3.

Each one of the principal congruence subgroups $\Gamma_i$ is an open subgroup of $\mathscr{O}^1$. In the case of $M$, $\mathscr{O}^1$ is an open subgroup of $\mathrm{SL}_2(\mathbf{Q}_p)$ and, in the other case, $\mathscr{O}^1 = D^1$ and so $D^1$ is compact. With this discussion together with Theorem 4.2, we deduce:

COROLLARY 4.4. $\mathrm{SL}_2(\mathbf{Q}_p)$ *and* $D^1$ *are p-adic Lie groups of dimension* 3.

We next discuss the Lie algebras of these Lie groups. Let $A_0$ denote the subspace of pure quaternions of $A$. For $x, y \in A_0$ define $[x, y] = xy - yx$. Under this operation, $M_0$ and $D_0$ become three-dimensional $p$-adic Lie algebras, denoted $\mathscr{L}_M$ and $\mathscr{L}_D$ respectively, such that $[\mathscr{L}_M, \mathscr{L}_M] = \mathscr{L}_M$, $[\mathscr{L}_D, \mathscr{L}_D] = \mathscr{L}_D$. For any basis $e_1, e_2, e_3$ of such a Lie algebra $\mathscr{L}$, the set $f_1 = [e_2, e_3]$, $f_2 = [e_3, e_1]$, $f_3 = [e_1, e_2]$ is also a basis and the matrix $M(\mathscr{L}) = [a_{ij}]$ where $f_i = \Sigma a_{ij} e_j$ is symmetric. In the cases above, starting with the standard basis from the quaternion algebra $\{i, j, ij\}$, the matrices are

$$M(\mathscr{L}_M) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad M(\mathscr{L}_D) = \begin{pmatrix} -2p & 0 & 0 \\ 0 & -2u & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

THEOREM 4.5. *Let* $\mathscr{L}$ *be a Lie Algebra over* $\mathbf{Q}_p$ *of dimension* $\leq 3$. *Then* $\mathscr{L}$ *is solvable or* $\mathscr{L} \cong \mathscr{L}_M$ *or* $\mathscr{L}_D$, *and further* $\mathscr{L}_M \not\cong \mathscr{L}_D$.

*Proof.* If $\dim \mathscr{L} \leq 2$ or $\mathscr{L} \neq [\mathscr{L}, \mathscr{L}]$ then $\mathscr{L}$ is solvable (see [9, Chap. 1; 10]). Thus we can assume that $\mathscr{L}$ has dimension 3 and that $\mathscr{L} = [\mathscr{L}, \mathscr{L}]$. Then choosing a basis as above, we obtain a $3 \times 3$ symmetric matrix $M(\mathscr{L})$ over $\mathbf{Q}_p$. Now given Lie algebras $\mathscr{L}_1$ and $\mathscr{L}_2$ we have that $\mathscr{L}_1$ and $\mathscr{L}_2$ are isomorphic if and only if there exists $x \in \mathbf{Q}_p^*$ and $N \in \mathrm{GL}_3(\mathbf{Q}_p)$ such that

$$M(\mathscr{L}_1) = xN^t M(\mathscr{L}_2) N$$

(see [9, Chap. 1]). Denote the equivalence relation $\sim_1$.

The symmetric matrices $M(\mathscr{L})$ determine ternary quadratic forms and the corresponding quadratic spaces $Q(\mathscr{L}_1), Q(\mathscr{L}_2)$ are isometric if and only if there exists $N \in \mathrm{GL}_3(\mathbf{Q}_p)$ such that

$$M(\mathscr{L}_1) = N^t M(\mathscr{L}_2) N.$$

Denote this equivalence relation $\sim_2$. The spaces $Q(\mathscr{L}_1), Q(\mathscr{L}_2)$ are isometric if and only if $d(\mathscr{L}_1) = d(\mathscr{L}_2)$ and $s(\mathscr{L}_1) = s(\mathscr{L}_2)$, where $d(\mathscr{L})$ denotes the determinant of the matrix $M(\mathscr{L})$ as an element of $\mathbf{Q}_p^*/(\mathbf{Q}_p^*)^2$ and $s(\mathscr{L})$ the Hasse invariant of $M(\mathscr{L})$, which is this case is a quaternion algebra. (See [12, p. 124]).

Note that $d(\mathscr{L}_M) = 2$, $d(\mathscr{L}_D) = 2up$, $s(\mathscr{L}_M) = M$, and $s(\mathscr{L}_D) = D$. So for each $\mathscr{L}$, $s(\mathscr{L}) = M$ or $D$. Furthermore, for every $\mathscr{L}' \sim_1 \mathscr{L}$, $s(\mathscr{L}') = s(\mathscr{L})$ and so $\mathscr{L}_M \not\cong \mathscr{L}_D$. Now we can find $\mathscr{L}' \sim_1 \mathscr{L}$ such that $d(\mathscr{L}') = 2$ or $2up$. Thus $\mathscr{L}' \sim_2 \mathscr{L}_M$ of $\mathscr{L}_D$ and hence $\mathscr{L} \sim_1 \mathscr{L}_M$ or $\mathscr{L}_D$. ∎

A global chart can be defined on $\Gamma_1$ using the log map. For

$$\log(1+p\alpha) = \Sigma(-1)^{n+1}\frac{(p\alpha)^n}{n}$$

converges to an element of $p\mathscr{O}$ for $\alpha \in \mathscr{O}$. Denote the reduced norm of $A$ by $n_A$. For $1+p\alpha \in \mathscr{O}^1$, $n_A(1+p\alpha) = (1+p\alpha)(1+p\bar{\alpha}) = 1$ and so $\log(1+p\alpha)$ has trace zero. Hence $\log \Gamma_1 \subset p\mathscr{O} \cap A_0$, where $A_0$ is the three-dimensional subspace of pure quaternions.

Now applying the exponential map to pure quaternions in $p\mathscr{O}$ gives elements of $\Gamma_1$, so that $p\mathscr{O} \cap A_0$ is the precise image of $\Gamma_1$ under the log map. Since $\mathscr{L}_M \cong (p\mathscr{O} \cap M_0) \otimes \mathbf{Q}_p$ and similarly for $\mathscr{L}_D$, these are the Lie algebras of the groups $\mathrm{SL}_2(\mathbf{Q}_p)$ and $D^1$ respectively. Thus, summarizing this section, we have

THEOREM 4.6. *Let $G$ be a $p$-adic Lie group of dimension* 3 *with Lie algebra $\mathscr{L}$. If $\mathscr{L}$ is not solvable, then $\mathscr{L}$ is isomorphic to the Lie algebra of exactly one of $\mathscr{L}_M, \mathscr{L}_D$, the Lie algebras of* $\mathrm{SL}_2(\mathbf{Q}_p)$, *and the compact group $D^1$ respectively*.

## 5. PROOF

Here we shall prove the following theorem, from which Theorem 1.2 follows. As mentioned in the Introduction, this is well known for geoemetric reasons in most cases.

THEOREM 5.1. *Every co-compact Fuchsian group which is not a triangle group splits as a free product with amalgamation.*

By Lemma 3.2, there is a representative of $\sigma$ in $\mathrm{PSL}_2(K)$ which is non-integral at $\mathscr{P}$ and for which $p = \mathscr{P} \cap \mathbf{Z}$ is an odd prime that splits completely in $K$. By, for example, [11] or [14], $K_{\mathscr{P}} \cong \mathbf{Q}_p$.

It will be convenient to work with a representative in $\mathrm{SL}_2(K)$, e.g., as at (2), which we denote by $F$, and let $i(F)$ denote the image in $\mathrm{SL}_2(\mathbf{Q}_p)$, under the map induced by completion at $\mathscr{P}$ and the above isomorphism. We now intend to show that $i(F)$ has property (D) and apply Theorem 2.4. Since $\mathrm{SL}_2(\mathbf{Q}_p)$ is a closed subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$, it clearly suffices to show that $i(F)$ is dense in $\mathrm{SL}_2(\mathbf{Q}_p)$. From this, together with the remarks following Theorem 2.4, the proof of Theorem 5.1 will be complete.

Let $\mathrm{cl}(F)$ denote the closure of $i(F)$ in $\mathrm{SL}_2(\mathbf{Q}_p)$, which, by a result of Cartan, cf. [18, p. 155], as a closed subgroup of the $p$-adic Lie group $SL(2, \mathbf{Q}_p)$, is a $p$-adic Lie group.

LEMMA 5.2. *The Lie algebra of* cl$(F)$ *is not solvable.*

*Proof.* Suppose that the Lie algebra of cl$(F)$ were solvable. Then cl$(F)$ would contain a solvable open subgroup [4, Chap. 3].

First note that $i(F)$ is not discrete. For if it were, then every torsion-free subgroup of finite index would be discrete and hence free [17, p. 82; 8]. But $i(F)$ contains subgroups of finite index which are isomorphic to torsion-free co-compact Fuchsian groups. Thus $S \cap i(F)$ is a non-trivial solvable subgroup of $i(F)$. Since every solvable subgroup of a co-compact Fuchsian group is cyclic, it follows, by considering subgroups of finite index, that $S \cap i(F) = Z$ or $Z \oplus \langle -I \rangle$, where $Z = \langle z \rangle$ is infinite cyclic.

Since $i(F)$ is dense in cl$(F)$ and $S$ is an open subgroup, $S \cap i(F)$ is dense in $S$ and so $Z$ is dense in $S$. Let $w \in i(F)$ be an element of infinite order which does not commute with $z$. Let $U = S \cap wSw^{-1}$ which is an open solvable subgroup. Since $Z$ is dense in $S$, $wZw^{-1}$ is dense in $wSw^{-1}$. Thus there exist powers $z^n$, $wz^m w^{-1}$ which lie in $U$. But, for large enough $m$ and $n$, these elements generate a non-abelian free group $N$ (e.g., [13]). Thus $N$ cannot lie in $U$ and the Lie algebra of cl$(F)$ cannot be solvable. ∎

Now recall that $i(F)$ was constructed to have a non-integral trace. Thus cl$(F)$ cannot be compact. For, as a compact subgroup of $\mathrm{SL}_2(\mathbf{Q}_p)$, it would fix a vertex of the tree $T$ described in Section 2, and so by Lemma 2.2 would be conjugate to a group with $p$-adic integral traces. Hence we have

COROLLARY 5.3. cl$(F)$ *is a non-compact Lie group*

From Lemma 3.3, there is an element $g \in i(F)$ of the form $\begin{pmatrix} y & 0 \\ by + cy^{-1} & y^{-1} \end{pmatrix}$, where $y, b, c \in \mathbf{Q}_p$. By taking $h$ to be $\begin{pmatrix} y & 0 \\ d & y^{-1} \end{pmatrix}$ where $d = -(by + cy^{-1})/(y^2 - 1)$ we obtain that $hgh^{-1}$ is diagonal and has trace of the form $a/p$ where $a$ is a $p$-adic unit. Thus we shall assume without loss of generality that cl$(F)$ contains the diagonal element $g = \mathrm{diag}\{y, y^{-1}\}$ of a non-integral trace. We are now in a position to complete the proof.

THEOREM 5.4. cl$(F) = \mathrm{SL}_2(\mathbf{Q}_p)$.

*Proof.* Since cl$(F)$ is a $p$-adic Lie subgroup of $\mathrm{SL}_2(\mathbf{Q}_p)$, it has dimension $\leq 3$. Hence the Lie algebra of cl$(F)$ is a Lie subalgebra of $\mathscr{L}_M$ and therefore Theorems 4.5 and 4.6, Lemma 5.2, and Corollary 5.3 show that the Lie algebra of cl$(F)$ is isomorphic to $\mathscr{L}_M$.

By Theorem 4.2 cl$(F)$ contains an open uniform pro-$p$ subgroup which is compact, and thus via its action on the tree T, there is an $x \in \mathrm{GL}_2(\mathbf{Q}_p)$ such that $x\,\mathrm{cl}(F)x^{-1}$ contains a subgroup $\Gamma$ of finite index in $\mathrm{SL}_2(\mathbf{Z}_p)$. In particular, as $\Gamma$ is open it contains one of the fundamental neighborhoods of the identity, namely one of the principal congruence subgroups $\Gamma_j$ of

Section 4. By increasing $j$ if necessary, we can ensure that $x^{-1}\Gamma_j x \subset \mathrm{cl}(F) \cap \mathrm{SL}_2(\mathbf{Z}_p)$. Since $x^{-1}\Gamma_j x$ is open, it therefore follows (as above) that there is an $i$ such that $\Gamma_i \subset x^{-1}\Gamma_j x$, and hence $\Gamma_i \subset \mathrm{cl}(F)$.

We therefore conclude that $\mathrm{cl}(F)$ contains some $\Gamma_i$, and from the discussion prior to the theorem, the diagonal element $g = \left(\begin{smallmatrix} y & 0 \\ 0 & y^{-1}\end{smallmatrix}\right)$ where $y = v/p$ with $v$ a $p$-adic unit. Now $\mathrm{SL}_2(\mathbf{Q}_p)$ is generated by the subgroups

$$U = \left\{ \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \middle| q \in \mathbf{Q}_p \right\} \qquad \text{and} \qquad L = \left\{ \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \middle| q \in \mathbf{Q}_p \right\}$$

[16]. Now

$$U \cap \Gamma_i = \left\{ \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \middle| \nu_p(q) \geq i \right\} = U_i.$$

Let $\left(\begin{smallmatrix} 1 & q \\ 0 & 1\end{smallmatrix}\right) \in U$ with $q = wp^s$ where $w$ is a unit and $s \in \mathbf{Z}$. Choosing $m$ such that $-2m + s \geq i$, it follows that $g^m \left(\begin{smallmatrix} 1 & q \\ 0 & 1\end{smallmatrix}\right) g^{-m} \in U_i$ and so $U \subset \langle U_i, g \rangle$. In the same way $L \subset \langle L_i, g \rangle$. Thus $\mathrm{cl}(F) = \mathrm{SL}_2(\mathbf{Q}_p)$. ∎

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