Determining hyperbolic 3–manifolds by their surfaces

D. B. McReynolds[*] and A. W. Reid[†]

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Abstract

In this article, we prove that the commensurability class of a closed, orientable, hyperbolic 3–manifold is determined by the surface subgroups of its fundamental group. Moreover, we prove that there can be only finitely many closed, orientable, hyperbolic 3–manifolds that have the same set of surfaces.

1   Introduction

The geodesic length spectrum of a Riemannian manifold M is a basic invariant that has been well-studied due to its connection with the geometric and analytic structure of M. For instance, when M has negative sectional curvature, there is a strong relationship between this spectrum and the eigenvalue spectrum of the Laplace–Beltrami operator (see [8],[9]), and the latter is well known to determine basic geometric/topological invariants like the dimension, volume, and total scalar curvature of M.

In this article, we focus on variations of the surface analog of the geodesic length spectrum for closed, orientable, hyperbolic 3–manifolds introduced by the authors in [20] (see also [13], [21], and [22]). We take this theme further and study *the full surface spectrum (or set)* of such manifolds (see §2 for definitions) which loosely takes into account all of the π_1–injective surface subgroups of the fundamental group of M. Our main result can be informally stated as follows (see Theorem 3.1 for the precise statement).

Theorem 1.1. *For any closed, orientable, hyperbolic 3–manifold M, there are at most finitely many non-isometric closed, orientable, hyperbolic 3–manifolds with the same surface set as M. Furthermore, all such manifolds are commensurable.*

For the eigenvalue or geodesic length spectra, many commensurability and finiteness results have been established. The second author [24] proved that isospectral (i.e. the same eigenvalue spectra) or length isospectral (i.e. the same geodesic length spectra) arithmetic hyperbolic 2–manifolds are commensurable. Chinburg–Hamilton–Long–Reid [7, Thm 1.1]

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proved an identical result for arithmetic hyperbolic 3–manifolds. Prasad–Rapinchuk [23, Thm 8.12] determined when these commensurability rigidity results hold for general, arithmetic, locally symmetric orbifolds, proving that in many settings the commensurability class of the manifold is determined by the eigenvalue or geodesic length spectra. It was already known that the commensurability class is not always determined by these spectra as Lubotzky–Samuel–Vishne [14, Thm 1] produced higher rank, arithmetic, locally symmetric incommensurable isospectral examples prior to the work of Prasad and Rapinchuk. In [20, Thm 1.1], the authors proved a result similar to Theorem 1.1. Namely, if M_1, M_2 are arithmetic hyperbolic 3–manifolds that contain a totally geodesic surface, and have the same set of totally geodesic surfaces, then they are commensurable. Meyer [21, Thm C] established a higher dimensional analog for certain classes of arithmetic hyperbolic n–manifolds. It is worth emphasizing that our present work differs from all the above works in one important and fundamental way. Namely, we do not impose an arithmetic assumption.

In [20, Thm 1.2], examples of non-isometric, closed, hyperbolic 3–manifolds with the same spectra of totally geodesic surfaces were constructed (see also [18, §5] and [19]). Those methods can be employed to also produce arbitrarily large finite sets of non-isometric closed hyperbolic 3–manifolds $\{M_j\}$ that pairwise have the same totally geodesic surface spectra (the spectra can be ensured to be infinite as well). However, it is unknown if an infinite set of such manifolds can exist. In particular, the totally geodesic surface analog of our finiteness result is unknown. Finally, for the full surface spectrum, there are no known examples of non-isometric hyperbolic 3–manifolds M_1, M_2 with the same full surface spectrum.

2   Notation and Preliminaries

Throughout, $M = \mathbf{H}^3/\Gamma$ will be a closed, orientable, hyperbolic 3–manifold and Σ_g will denote the closed orientable surface of genus g. It was proved by Thurston [28, Cor 8.8.6] that the number of Γ–conjugacy classes of subgroups of Γ isomorphic to $\pi_1(\Sigma_g)$ is finite. A breakthrough was provided by Kahn and Markovic [11, Thm 1.1] who proved that this number is non-zero for certain values of g. Furthermore, they then went onto provide estimates for these numbers in [12, Thm 1.1] (building on previous work of Masters [17, Thm 1.2]).

We need to refine this discussion somewhat. For each discrete, faithful representation $\rho\colon \pi_1(\Sigma_g) \to \mathrm{PSL}(2, \mathbf{C})$, we refer to the image Δ_ρ as a *Kleinian surface group*. For each Δ_ρ, let $\ell_\rho(M)$ denote the number of Γ–conjugacy classes of subgroups $\Delta < \Gamma$ that are $\mathrm{PSL}(2, \mathbf{C})$–conjugate to Δ_ρ. Typically the value of $\ell_\rho(M)$ will be zero (e.g. for those Δ_ρ that contain an element with transcendental trace), but for those that are non-zero we can define the full surface spectrum of M to be the collection of such pairs $(\Delta_\rho, \ell_\rho(M))$. Specifically, the *full surface spectrum* of M is the set $\mathcal{S}(M) = \{(\Delta_\rho, \ell_\rho(M)) : \ell_\rho(M) \neq 0\}$. Additionally, we define the *surface set* of M to be the set $\mathrm{S}(M) = \{\Delta_\rho : \ell_\rho(M) \neq 0\}$. The case when Δ_ρ is Fuchsian was studied in [20] and gives rise to an associated spectrum that we denote here by $\mathrm{S}_{Fuc}(M)$ and call the *genus spectrum*.

In this note, particular emphasis will be placed upon those Kleinian surface groups Δ_ρ corresponding to virtual fiber subgroups of Γ. By the work of Bonahon [4] and Thurston

[28] (and more generally the solution to the Tameness Conjecture by Agol [1] and Calegari–Gabai [6]), these virtual fiber subgroups of Γ are precisely those Δ_ρ that are finitely generated, geometrically infinite subgroups of Γ. Since being geometrically infinite depends only on Δ_ρ and not on the ambient group Γ, these surface subgroups provide an important subclass of surface subgroups that can be used to control the topology of 3–manifolds. For future reference, let us denote the associated spectrum for this subclass of surface subgroups by

$$\mathrm{S}_{vf}(M) = \{\Delta_\rho \in \mathrm{S}(M) \ : \Delta_\rho \text{ is a virtual fiber subgroup}\}.$$

Essential in our work is the groundbreaking work of Agol [2] and the aforementioned work of Kahn–Markovic [11]. We summarize their collective work in the following theorem.

Theorem 2.1. *Let $M = \mathbf{H}^3/\Gamma$ be a closed, orientable, hyperbolic 3–manifold. Then*

(a) $\mathrm{S}(M) \neq \emptyset$.

(b) $\mathrm{S}_{vf}(M) \neq \emptyset$.

(c) $\mathrm{S}_{vf}(M)$ *contains infinitely many elements F_ρ that are not commensurable and in particular have arbitrarily large genus.*

Proof: Given the preamble to the statement of the theorem, the only part that needs comment is (c). By [2] there is a finite sheeted cover $M_0 \to M$ such that $b_1(M_0) \geq 2$ and M_0 is fibered. In particular, by [29], M_0 is fibered in infinitely many different ways. Indeed, it follows from [29] that we can find fibered surfaces of arbitrarily large genus occuring as integral lattice points in the (open) cone over a top dimensional face of the Thurston norm ball. Since the degree of the cover $M_0 \to M$ is finite and the fibers have arbitrarily large genus, projecting these fibers back to M provides infinitely many incommensurable virtual fibers. $\square$

3 Proof of Theorem 1.1

We now state the precise version of Theorem 1.1 that we will prove in this section.

Theorem 3.1. *If M is a closed, orientable, hyperbolic 3–manifold, then the set*

$$\mathcal{S}_M = \{N \ : \mathcal{S}(M) = \mathcal{S}(N)\}$$

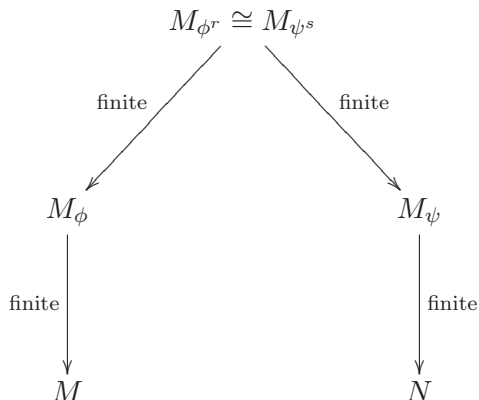*is finite. Moreover, if $N \in \mathcal{S}_M$, then M, N are commensurable.*

As noted above, since being a virtual fiber depends only on Δ_ρ and not on the ambient manifolds, if $\mathcal{S}(M) = \mathcal{S}(N)$, then $\mathrm{S}_{vf}(M) = \mathrm{S}_{vf}(N)$. In particular, to prove Theorem 3.1, it suffices to prove the following result.

Theorem 3.2. *If M is a closed, orientable, hyperbolic 3–manifold, then the set*

$$\mathcal{S}_{M,vf} = \{N \ : \ \mathrm{S}_{vf}(M) = \mathrm{S}_{vf}(N)\}$$

is finite. Moreover, if $N \in \mathcal{S}_{M,vf}$, then M, N are commensurable.

Proof of Theorem 3.2: We first prove that if $S_{vf}(M) = S_{vf}(N)$, then M, N are commensurable. To that end, let $\Delta = \Delta_\rho$ denote a common virtual fiber subgroup and set g to be the genus of Δ. Since Δ is a virtual fiber, we can find pseudo-Anosov maps $\phi, \psi \colon \Sigma_g \longrightarrow \Sigma_g$ so that $M_\phi \to M$, $M_\psi \to N$ are finite sheeted covers and $\pi_1(M_\phi), \pi_1(M_\psi)$ have a common fiber group Δ. Associated to the fiber group Δ is a unique pair of ending laminations $\nu^\pm$ in the projective measured lamination space of Σ_g which are left invariant by ϕ, ψ (see [4]). As a result, there exist integers r, s such that the mapping classes ϕ, ψ satisfy $\phi^r = \psi^s$. Consequently, the bundles M_{ϕ^r} and M_{ψ^s} are isometric. In particular, we have

$$M_{\phi^r} \cong M_{\psi^s}$$

finite $\quad\quad\quad\quad$ finite

$$M_\phi \quad\quad\quad\quad\quad\quad M_\psi$$

finite $\quad\quad\quad\quad\quad\quad\quad$ finite

$$M \quad\quad\quad\quad\quad\quad\quad\quad N$$

and thus conclude that M, N are commensurable.

It remains to establish the finiteness of $\mathcal{S}_{M,vf}$. We will argue by contradiction, and to that end, we assume that there are infinitely many non-isometric $M_i = \mathbf{H}^3/\Gamma_i$, $i = 1, 2, \dots$ with $S_{vf}(M) = S_{vf}(M_i)$ for all i. We will prove that for $i \geq i_0$, the groups Γ_i have uniformly bounded rank. We will then show that for an even larger i_1, the groups Γ_i for $i \geq i_1$ must have rank larger than this uniform bound. Towards that goal, we first assert that the volumes of the manifolds M_i must be unbounded. Specifically, we have the following general lemma.

Lemma 3.3. *The set of volumes for any infinite set $\{M_i\}$ of commensurable, finite volume, hyperbolic 3–manifolds is unbounded.*

Proof: We split into two cases depending on whether the manifolds are arithmetic or not. Note that since arithmeticity is a commensurability invariant, either all of the M_i are arithmetic or all of the M_i are non-arithmetic. If the M_i are arithmetic, the assertion follows from work of Borel [5] since there are only finitely many arithmetic hyperbolic 3–manifolds of bounded volume. If the M_i are non-arithmetic, by work of Margulis [16], there is a unique maximal lattice in the common commensurability class that contains all of the Γ_i as finite index subgroups. In particular, all the M_i cover the fixed closed hyperbolic 3–orbifold Q associated to this unique maximal lattice. Since Q has only finitely many degree d covers for any d, the covering degrees must go to infinity. Consequently, the volumes cannot be bounded in this case either. $\square$

We further note that since the manifolds M_i are all commensurable, there is also a uniform lower bound of the injectivity radii of the M_i. This is easily proved using the

arithmetic/non-arithmetic dichotomy once again. For future reference, we denote this lower bound by s. Thus we can assume that we have a sequence of manifolds M_i with injectivity radius s and whose volumes get arbitrarily large. We now show how to use this to bound the ranks of the groups Γ_i for i sufficiently large.

Towards that goal, set Δ_0 to be a common, minimal genus, virtual, fiber group in Γ_i, and set g to be this common, minimal genus. In order to control the ranks of the groups Γ_i, we will utilize a quantitative virtual fibering result of Soma [25, Thm 0.5]. To state his result, let $\mathrm{InjRad}(M), \mathrm{Vol}(M)$ denote the injectivity radius and volume of M, respectively, and set $d_1(g,s) = \frac{4s(g-1)}{\sinh^2(s/2)}$.

Theorem 3.4 (Soma). *If M is a closed, orientable, hyperbolic 3–manifold with*

$$\mathrm{InjRad}(M) \geq s \text{ and } \mathrm{Vol}(M) \geq 2\pi d_1(g,s)\sinh^2(d_1(g,s)+1),$$

then any immersed virtual fiber in M of genus g is embedded.

Theorem 3.4 in tandem with the above conditions on $\mathrm{InjRad}(M_i), \mathrm{Vol}(M)$ implies that there is $i_{g,s} \in \mathbf{N}$ such that if $i \geq i_{g,s}$, the virtual fiber group Δ_0 corresponds to an embedded incompressible surface of genus g in M_i. This incompressible surface greatly limits the structural possibilities for the manifolds M_i. Specifically, M_i is either a fiber bundle over the circle with fiber group Δ_0, or M_i is the union of two twisted I–bundles. Moreover, in the latter case, we have a double cover $N_i \to M_i$ such that N_i is a fiber bundle with fiber group Δ_0 (see [10, Ch 11]).

We now leverage the above fiber bundle structure to obtain bounds for the rank of Γ_i for i sufficiently large. The rank of Γ_i will be denoted by $\mathrm{Rank}(\Gamma_i)$.

Lemma 3.5. *There exists $i_0 \geq i_{g,s}$ such that if $i \geq i_0$, then $g + 1 \leq \mathrm{Rank}(\Gamma_i) \leq 2g + 2$.*

Proof: We assume throughout that $i \geq i_{g,s}$. Let $\mathcal{I}_1$ to be the set of $i \geq i_{g,s}$ such that M_i is a fiber bundle with fiber group Δ_0 and let $\mathcal{I}_2$ to be the set of $i \geq i_{g,s}$ such that M_i is double covered by N_i where N_i is a fiber bundle with fiber group Δ_0. We first consider $\{M_i\}_{i \in \mathcal{I}_1}$. We know from the proof of the commensurability invariance of S_{vf} that each M_i must have the form $M_{\phi^{r_i}}$ for some pseudo-Anosov element ϕ. Applying Souto [26, Thm 1], there exist $i' \in \mathbf{N}$ such that $\mathrm{Rank}(\Gamma_i) = 2g+1$ for all $i \geq i'$. Next, we consider $\{M_i\}_{i \in \mathcal{I}_2}$ and apply the above argument to N_i. We obtain $i'' \in \mathbf{N}$ such that $\mathrm{Rank}(\pi_1(N_i)) = 2g + 1$ for all $i \geq i''$. As N_i is a double cover of M_i, we can adjoin one element to $\pi_1(N_i)$ to generate $\pi_1(M_i)$. Therefore, $\mathrm{Rank}(\Gamma_i) \leq 2g + 2$ for all $i \geq i''$.

Now, set $i_0 = \max\{i', i''\}$ and note that $\mathrm{Rank}(\Gamma_i) \leq 2g + 2$ for all $i \geq i_0$. For the lower bound, by the Nielsen–Schreier, we have $\mathrm{Rank}(\pi_1(N_i)) \leq 2\mathrm{Rank}(\Gamma_i) - 1$ for all $i \geq i_0$ and $i \in \mathcal{I}_2$. In particular, $g + 1 \leq \mathrm{Rank}(\Gamma_i)$ for all $i \geq i_0$. $\square$

We now use Lemma 3.5 to complete the proof of Theorem 3.2. By Theorem 2.1 (c), we can find incommensurable virtual fiber subgroups of arbitrarily large genus. Choosing a virtual fiber subgroup Δ_1 of genus g_1 with $2g + 2 < g_1 + 1$ and repeating the above argument, we obtain an integer $i_1 \geq i_{g_1,s}$ such that $g_1 + 1 \leq \mathrm{Rank}(\Gamma_i)$ for all $i \geq i_1$. For all $i \geq \max\{i_0, i_1\}$, we must have $g_1 + 1 \leq \mathrm{Rank}(\Gamma_i) \leq 2g + 2 < g_1 + 1$, a contradiction.

Hence $\mathcal{S}_{M,vf}$ is finite as required. $\square$

Remarks:(1) In the proof of Lemma 3.5 we could also have used [3] for both the bundle case and the union of two twisted I-bundles. However, the setting of [26] is more appropriate in this case (i.e. commensurable manifolds), and only a mild extension is needed for us to handle the union of two twisted I-bundles. Hence the reason for not using [3] in this case. In §4, we will need to use [3].

(2) As noted in the introduction, we do not know if there exists a pair of non-isometric, closed, orientable, hyperbolic 3–manifolds M_1, M_2 with $\mathcal{S}(M_1) = \mathcal{S}(M_2)$. Since being either a virtual fiber or Fuchsian depends only on Δ_ρ and not the ambient manifold, such a pair would also satisfy both $\mathrm{S}_{vf}(M_1) = \mathrm{S}_{vf}(M_2)$, $\mathrm{S}_{Fuc}(M_1) = \mathrm{S}_{Fuc}(M_2)$. Examples where the latter equality holds were constructed in [20] using a variation of the method of Sunada [27] for constructing isospectral and length isospectral manifolds. That method does not seem well-suited for also arranging equality between virtual fibers. As with the full spectrum, we do not presently know if there exists a pair of non-isometric, closed, hyperbolic 3–manifolds M_1, M_2 with $\mathrm{S}_{vf}(M_1) = \mathrm{S}_{vf}(M_2)$.

4 A conjectural strengthening for arithmetic hyperbolic 3–manifolds

In this section we deal with closed, arithmetic, hyperbolic 3–manifolds, and prove a stronger result (conjecturally) that involves only topological data. We refer the reader to [15] for background on arithmetic hyperbolic 3–manifolds. Let us define *the topological virtual fiber set* of M to be the set

$$\mathrm{S}_{tvf}(M) = \{\Delta \; : \Delta \text{ is isomorphic to a virtual fiber subgroup}\}.$$

Our strengthening relies on the following conjecture often referred to as the *short geodesic conjecture*.

Conjecture 1 (Short Geodesic Conjecture). *Let M be a closed, orientable, arithmetic, hyperbolic 3–manifold. Then there is a constant $C > 0$ (independent of M) so that the length of the shortest closed geodesic in M is at least C.*

Assuming this conjecture, we establish the following result.

Theorem 4.1. *Assuming Conjecture 1 there are at most finitely many closed orientable arithmetic hyperbolic 3–manifolds $M_1, M_2 \ldots M_n$ so that $\mathrm{S}_{tvf}(M_i) = \mathrm{S}_{tvf}(M_j)$ for each i, j.*

Proof: The proof of Theorem 4.1 is similar to the proof of Theorem 3.2 and is done by contradiction. If there is an infinite sequence of such manifolds M_i, by Borel [5] their volumes are unbounded and Conjecture 1 implies that the injectivity radii are bounded from below. Choosing a minimal genus (topological) virtual fiber in each M_i and applying Theorem 3.4, it follows that for sufficiently large i, M_i is either a genus g fiber bundle or

a union of two twisted I–bundles which is double covered by a genus g fiber bundle. We now apply Biringer's extension of [26], namely [3, Thms 1.1, 5.2]. That allows us to get control of the rank as in the proof of Lemma 3.5, and in particular, following the argument in the proof of Lemma 3.5 leads to a similar contradiction on ranks as used in the proof of Theorem 3.2. $\square$

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Department of Mathematics
Purdue University
West Lafayette, IN 47906
email: `dmcreyno@purdue.edu`

Department of Mathematics
University of Texas
Austin, TX 78712
email: `areid@math.utexas.edu`