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BOUNDS ON THE AVERAGE VELOCITY OF A RIGID BODY IN A STOKES FLUID*

O. GONZALEZ†

Abstract. Bounds are established for the average velocity per unit force for the motion of a rigid body in a viscous fluid modeled with the Stokes equations. The translational velocity in the direction of an imposed force is considered in a fluid that is at rest at infinity, and the average is taken over all orientations of the force, with the orientations being uniform on the unit sphere. This average velocity for an arbitrary rigid body is shown to be bounded above and below by two simple, characteristic inverse distances associated with the surface of the body. Whereas the lower bound is implied by classic comparison results, the upper bound is of a different character and does not rely on any notion of an enclosing surface as in classic results, and may be useful in making comparisons between two bodies when neither can be enclosed by the other. In contrast to previous studies based on partial differential equations for the primary field variables, the bounds are established using a boundary integral formulation and an associated constrained extremum principle for the potential density. The results derived here additionally imply a purely geometric inequality for surfaces that may be of independent interest.

Key words. Stokes equations, mobility matrices, layer potentials, extremum principles

AMS subject classifications. 76D07, 31B10, 35Q30, 74F10

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1. Introduction.

The problem of characterizing the motion of a rigid body in a viscous fluid is important in many areas of science and engineering. This problem arises, for example, in the study of the rheology of colloidal suspensions [13, 22], the study of experimental methods for probing the structure of macromolecules such as proteins and DNA [5], and the modeling of various devices for the separation and manipulation of particles in microfluidic systems [23], including the optimal design of microrobots in connection with targeted drug delivery [20]. In applications such as these, the Stokes equations provide a first approximation of the more general Navier–Stokes equations in situations where the flow is nearly steady and slow, and has small velocity gradients, so that inertial effects can be ignored.

When the motion of a rigid body in a viscous fluid is modeled with the Stokes equations there is a linear relation between the external force and torque on the body and its translational and rotational velocities. For instance, when a given reference point in the body is subject to a unit force, the body will respond with a characteristic translational and rotational velocity about that point, depending on the orientation of the force. The component of the translational velocity in the direction of the force is a natural quantity to consider, and the average of this component over all possible orientations of the force, referred to as the average velocity per unit force, is of special interest. This average velocity plays a central role in the convective and diffusive transport of rigid particles in viscous fluids [1, 2, 3, 4, 10, 11]. Indeed, when suitably scaled, the average velocity per unit force can be identified with the translational diffusion coefficient of the particle.

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†Department of Mathematics, The University of Texas at Austin, Austin, TX 78712 (og@math.utexas.edu).

Here we show that the average velocity per unit force of an arbitrary rigid body is bounded above and below by two simple, characteristic inverse distances associated with the surface of the body. Specifically, when suitably scaled, we show that the average velocity is bounded below by twice the minimum inverse distance and above by the average inverse distance between pairs of points on the surface. These bounds are obtained when the centroid of the surface is used as the reference point. Analogous bounds can be obtained for other choices of reference point at the expense of simplicity. As a corollary, our bounds imply that, for an arbitrary surface, the average inverse distance between pairs of points is always at least twice the minimum inverse distance, with equality achieved in the case of a sphere. This purely geometric inequality gives rise to a number of natural calculus of variations questions and may be of independent interest.

The lower bound derived here is consistent with well-known comparison results pertaining to the viscous energy dissipation and drag on bodies in Stokes flow, when one body surface can be completely enclosed by another [16, 22]. In contrast, the upper bound derived here is of a different character; it is not related to any notion of an enclosing or enclosed surface and does not seem to be well-known. The bounds are of intrinsic mathematical interest within the theory of slow viscous flow and moreover may be of use in some applications, for instance, in making comparisons between two bodies when neither can be enclosed by the other. The bounds also suggest how certain modifications of a body can ultimately affect its average velocity: decreasing the average inverse distance between pairs of points on the surface of a body can be expected to ultimately lower the average velocity, while increasing the minimum inverse distance between pairs of points can be expected to ultimately raise the average velocity.

Our proof of the bounds is based entirely on classic potential theory for the Stokes equations [17, 24, 26, 27]. We assume only that the bounding surface of the body satisfies the usual Lyapunov conditions of potential theory [12] and work within the space of continuous density functions on this surface. Using symmetry and positivity properties of the single-layer potentials, we establish two extremum principles, one of which leads to the lower bound and the other of which leads to the upper bound. Although the subject of extremum principles for Stokes flows has been well-studied from the point of view of partial differential equations [16, 18, 19, 21, 29], here we consider such principles from the point of view of boundary integral equations and obtain bounds of a different character than previously reported. Moreover, our approach leads to a constrained extremum principle for the single-layer potential which appears to be not well-known and may be useful in other contexts.

2. Statement of main result.

Here we outline the Stokes equations and introduce the flow quantities and notation that are needed to state the main result. For further background, see [13, 22, 24].

2.1. Stokes equations, domain.

We consider the slow motion of a body in an incompressible viscous fluid in three-dimensional space. We denote the body domain by $D^- \subset \mathbb{R}^3$, the fluid domain by $D^+ \subset \mathbb{R}^3$, and the boundary between them by $\Gamma \subset \mathbb{R}^3$. The Stokes equations that describe the velocity field $u^+ : D^+ \to \mathbb{R}^3$ and pressure field $p^+ : D^+ \to \mathbb{R}$ of the fluid flow around the body are, in nondimensional form,

$$(2.1) \qquad \begin{aligned} u^+_{i,jj}(x) &= p^+_{,i}(x), \\ u^+_{i,i}(x) &= 0, \end{aligned} \qquad \text{or} \qquad \begin{aligned} \Delta u^+(x) &= \nabla p^+(x), \\ \nabla \cdot u^+(x) &= 0, \end{aligned} \qquad x \in D^+.$$

Equation $(2.1)_1$ is the local balance law of linear momentum for the fluid in the absence of an external force field and $(2.1)_2$ is the local incompressibility constraint. We assume that $D^- \cup \Gamma \cup D^+$ fills all of three-dimensional space, that D^- and D^+ are both open and connected, and that D^- is bounded. Moreover, we assume that Γ is closed and bounded and additionally is a Lyapunov surface [12]. The assumption of connectedness for D^- and D^+, and hence also Γ, is made for purposes of simplicity; the results outlined herein could be extended to more general cases.

Unless mentioned otherwise, all vector quantities are referred to a single frame and indices take values from one to three. Moreover, we use the usual conventions that a pair of repeated indices implies summation and that indices appearing after a comma denote partial derivatives. We assume here and throughout that all quantities have been nondimensionalized using a characteristic length scale $\ell > 0$, velocity scale $\vartheta > 0$, and force scale $\mu\vartheta\ell > 0$, where μ is the absolute viscosity of the fluid. The dimensional quantities corresponding to $\{x, u^+, p^+\}$ are $\{\ell x, \vartheta u^+, \mu\vartheta\ell^{-1}p^+\}$.

2.2. Boundary, decay conditions. For a given vector field $v : \Gamma \to \mathbb{R}^3$ we consider the Dirichlet boundary conditions

$$\begin{aligned} u_i^+(x) &= v_i(x), && x \in \Gamma, \\ u_i^+(x), p^+(x) &\to 0, && |x| \to \infty. \end{aligned} \tag{2.2}$$

Equation $(2.2)_1$ is the no-slip condition which states that the fluid and body velocities coincide at each point of the boundary, and $(2.2)_2$ is an asymptotic condition which is consistent with the fluid being at rest at infinity.

As will be outlined later, the stated assumptions on D^+ and Γ are sufficient to guarantee that, for any continuous boundary data v, the exterior Stokes system defined by (2.1) and (2.2) has a unique solution (u^+, p^+) among fields with the following decay properties [8, 15, 24]:

$$u_i^+(x) = O(|x|^{-1}), \quad u_{i,j}^+(x) = O(|x|^{-2}), \quad p^+(x) = O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty. \tag{2.3}$$

The solution (u^+, p^+) is smooth in D^+, but may possess only a finite number of bounded derivatives in $D^+ \cup \Gamma$ depending on the precise smoothness of Γ and v. Throughout our developments, we use the notation in $(2.2)_2$ as an abbreviation of the decay conditions in (2.3).

While we identify the domain D^- with an arbitrary material body, we can always consider an interior Stokes problem for fields (u^-, p^-) in D^- with Dirichlet data v on Γ. This interior problem is a natural mathematical companion to the exterior problem and arises in certain discussions, even though it may not be of any direct physical relevance. Here we note that solutions for the interior problem exist for continuous data v provided the solvability condition $\int_\Gamma v \cdot \nu \, dA_x = 0$ is met [24], where ν denotes the unit outward normal field on Γ and dA_x denotes an infinitesimal area element at $x \in \Gamma$. In this case, u^- is unique whereas p^- is unique only up to an additive constant. We remark that the exterior problem has no such solvability condition.

When the body domain D^- is in rigid-body motion, the vector field v in $(2.2)_1$ takes the general form

$$v_i(x) = V_i + \varepsilon_{ijk}\Omega_j(x_k - c_k) \qquad \text{or} \qquad v(x) = V + \Omega \times (x - c), \tag{2.4}$$

where ε_{ijk} is the standard permutation symbol. Here $V \in \mathbb{R}^3$ is the linear velocity of a given reference point $c \in \mathbb{R}^3$, and $\Omega \in \mathbb{R}^3$ is the angular velocity of the body.

One choice of reference point that will be of interest is the centroid $\overline{x}$ of the bounding surface Γ of D^-, defined by

$$\overline{x} = \frac{\int_\Gamma x \, dA_x}{\int_\Gamma \, dA_x}. \tag{2.5}$$

2.3. Stress, traction, loads. The stress field associated with a velocity-pressure pair (u^+, p^+) is a function $\sigma^+ : D^+ \to \mathbb{R}^{3\times 3}$ defined by

$$\sigma^+_{ij}(x) = -p^+(x)\delta_{ij} + u^+_{i,j}(x) + u^+_{j,i}(x), \tag{2.6}$$

where δ_{ij} is the standard Kronecker delta symbol. For each $x \in D^+$ the stress tensor σ^+ is symmetric in the sense that $\sigma^+_{ij} = \sigma^+_{ji}$. To highlight the dependence of σ^+ on the pair (u^+, p^+) we use the notation $\sigma^+[u^+, p^+]$.

The traction field $h^+ : \Gamma \to \mathbb{R}^3$ exerted by the exterior fluid on the body surface (force per unit area) is defined by

$$h^+_i(x) = \sigma^+_{ij}(x)\nu_j(x), \tag{2.7}$$

where $\nu : \Gamma \to \mathbb{R}^3$ denotes the outward unit normal field as before. The resultant force $F \in \mathbb{R}^3$ and torque $T \in \mathbb{R}^3$, about the reference point c, associated with the traction field are

$$F_i = \int_\Gamma h^+_i(x) \, dA_x, \qquad T_i = \int_\Gamma \varepsilon_{ijk}(x_j - c_j)h^+_k(x) \, dA_x. \tag{2.8}$$

The resultant loads (F, T) are purely hydrodynamic in the sense that they vanish when the body velocities (V, Ω) vanish; they are viscous drag loads.

2.4. Resistance, mobility matrices. When the body domain D^- is rigid and hence can only undergo rigid-body motion, we note that the boundary data must necessarily be of the form (2.4). In this case, linearity of (2.1)–(2.4) and (2.6)–(2.8) implies the existence of matrices $L_a \in \mathbb{R}^{3\times 3}$ $(a = 1, \ldots, 4)$ such that

$$F = -L_1 V - L_3 \Omega, \qquad T = -L_2 V - L_4 \Omega. \tag{2.9}$$

These matrices are called the hydrodynamic resistance matrices for the body [13, 14, 22]. They in general depend on the shape of the surface Γ and also, with the exception of L_1, the reference point c appearing in (2.4) and (2.8). As will be established later, the overall resistance matrix $L \in \mathbb{R}^{6\times 6}$ is symmetric and positive-definite, where

$$L = \begin{pmatrix} L_1 & L_3 \\ L_2 & L_4 \end{pmatrix}. \tag{2.10}$$

Throughout our developments we denote the inverse resistance matrix by $M \in \mathbb{R}^{6\times 6}$ so that

$$M = \begin{pmatrix} M_1 & M_3 \\ M_2 & M_4 \end{pmatrix} = \begin{pmatrix} L_1 & L_3 \\ L_2 & L_4 \end{pmatrix}^{-1}. \tag{2.11}$$

The block entries $M_a \in \mathbb{R}^{3\times 3}$ are called the mobility matrices. Similar to before, they in general depend on the shape of the surface Γ and also, with the exception of M_4, the reference point c. The overall mobility matrix M is also symmetric and positive-definite, and in view of (2.9) and (2.11) we have

$$V = -M_1 F - M_3 T, \qquad \Omega = -M_2 F - M_4 T. \tag{2.12}$$

2.5. Average velocity. When the body domain D^- is rigid and subject to external resultant loads $(F^{\text{ext}}, T^{\text{ext}})$, the slow motion of the body through the fluid is described by the load balance relations $F + F^{\text{ext}} = 0$ and $T + T^{\text{ext}} = 0$. Specifically, in the absence of an external torque, the rigid-body velocities generated by an external force of unit magnitude $F^{\text{ext}} = n$, $|n| = 1$, are

$$V = M_1 n, \qquad \Omega = M_2 n. \tag{2.13}$$

The component of the translational velocity V along the direction n of the force is given by

$$V \cdot n = n \cdot M_1 n. \tag{2.14}$$

The above quantity is a measure of the mobility of a body when subject to an external force; it is the velocity per unit force in the direction of the force. Of special interest is the average of this velocity over all directions n, given by

$$\mathcal{V} = \operatorname*{avg}_{n \in S^2} V \cdot n = \frac{1}{3}\operatorname{tr}(M_1), \tag{2.15}$$

where the expression in terms of the trace arises from direct integration of (2.14) over the unit sphere S^2. The average velocity $\mathcal{V}$ per unit force is an intrinsic property of a body and plays a central role in the theory of the convective and diffusive transport of rigid particles in a viscous fluid [1, 2, 3, 4, 10, 11]. For instance, a particle whose orientation is uniformly randomized due to thermal fluctuations would drift with average velocity $\mathcal{V}$ in response to a unit external force, which may be a body-type force due to gravity or an electric field, for example, or even an osmotic-type force due to a concentration gradient. Indeed, when scaled by the Boltzmann constant and the absolute temperature of the fluid, the average velocity $\mathcal{V}$ can be identified with the translational diffusion coefficient of a particle.

2.6. Main result. The average velocity $\mathcal{V}$ per unit force is an implicit function of the body surface Γ and the reference point c. Its numerical evaluation requires the construction of the linear map in (2.9) defined via the solution of the Stokes equations in the exterior domain around the surface, followed by an inversion of this map. Our main result shows that when the centroid of a surface is taken as the reference point, the average velocity is bounded above and below by two simple, characteristic inverse distances associated with the surface.

THEOREM 2.1. *Let Γ be a closed, bounded Lyapunov surface and let c be its centroid. Then the average velocity $\mathcal{V}$ per unit force satisfies*

$$2 \min_{x,y \in \Gamma} \frac{1}{|x-y|} \le 6\pi\mathcal{V} \le \operatorname*{avg}_{x,y \in \Gamma} \frac{1}{|x-y|}. \tag{2.16}$$

Thus the average velocity, up to a factor of 6π, is bounded below by twice the minimum inverse distance and above by the average inverse distance between pairs of points on the surface. The minimum inverse distance is simply the inverse of the diameter of the surface; it is achieved at the two most widely separated points. The average inverse distance is defined via averaging in the usual way, namely,

$$\operatorname*{avg}_{x,y \in \Gamma} \frac{1}{|x-y|} = \frac{\int_\Gamma \int_\Gamma \frac{1}{|x-y|}\, dA_x dA_y}{\int_\Gamma \int_\Gamma \, dA_x dA_y}. \tag{2.17}$$

From the above result we deduce that there is always a pair of distinct points $x_*, y_* \in \Gamma$ for which $6\pi\mathcal{V} = 1/|x_* - y_*|$. Hence the average velocity is inversely proportional to a certain characteristic size of the body. Note that the bounds in the above result are intrinsic in that they refer only to the surface Γ itself; no direct references are made to a second surface that encloses or is enclosed by Γ.

The lower bound in the above result is consistent with a classic comparison result [16, 22] pertaining to drag forces on rigid bodies in Stokes flow, when one body surface can be enclosed by another. Specifically, the lower bound given above is consistent with the classic result in the sense that Γ can always be enclosed by a sphere of the same diameter as Γ. Indeed, the proof given here for the lower bound establishes and exploits such a comparison principle for exterior Stokes flow around arbitrary Lyapunov surfaces. In contrast, the upper bound is of a significantly different character and does not seem to be well-known. Specifically, the upper bound is not directly related to any notion of an enclosing or enclosed surface, but is instead related to certain potential theoretic aspects of Stokes flow.

The elegant form of the bounds is a consequence of using the centroid as the reference point. As discussed in the proof, a more general form of the upper bound can be obtained for arbitrary choices of the reference point, but the resulting expression is rather complicated and its physical interpretation is less straightforward. The proof also reveals that an improved lower bound holds, namely,

$$\frac{2}{d} \le \frac{2}{d} + \frac{4|\delta|^2}{d^3} \le 6\pi\mathcal{V}, \tag{2.18}$$

where d is the diameter of Γ, and δ is the vector from the center of the bounding sphere to the centroid of Γ. For simplicity, we state our main result using the lower bound that is independent of the center of the bounding sphere, at the expense of some sharpness. The proof of the result also suggests that tighter bounds could naturally be obtained, but such bounds would not be as explicit.

Although motivated by hydrodynamic considerations, the above result implies a purely geometric inequality about surfaces. Specifically, for any closed, bounded Lyapunov surface Γ we have

$$2 \min_{x,y\in\Gamma} \frac{1}{|x-y|} \le \operatorname*{avg}_{x,y\in\Gamma} \frac{1}{|x-y|}, \tag{2.19}$$

which states that the average inverse distance is always at least twice the minimum inverse distance. While it is immediately evident that the average must be bounded below by the minimum, the factor of two is interesting. We remark that equality is achieved in the case of a sphere; the two sides of the inequality are both given by the inverse of the radius of the sphere. It is an open question about what other surfaces if any might also achieve equality, and also under what minimal assumptions the inequality would remain valid. It would also be of interest to consider surfaces which are extremals for the average inverse distance functional, subject to various constraints on diameter, enclosed volume, or surface area. Another sharper but perhaps less elegant geometric inequality can be obtained by combining (2.16) and (2.18).

Figure 1 presents bounds for four example surfaces. In addition to the bounds themselves, which depend on the overall size of a surface, we consider the scale invariant ratio of the upper to lower bound denoted by

$$\zeta = \left(\operatorname{avg}_{x,y\in\Gamma} \tfrac{1}{|x-y|}\right) / \left(2\min_{x,y\in\Gamma} \tfrac{1}{|x-y|}\right) \tag{2.20}$$

and also the actual average velocity $\mathcal{V}$. The average velocity is computed from a numerical boundary element technique described elsewhere [9, 25], whereas the upper

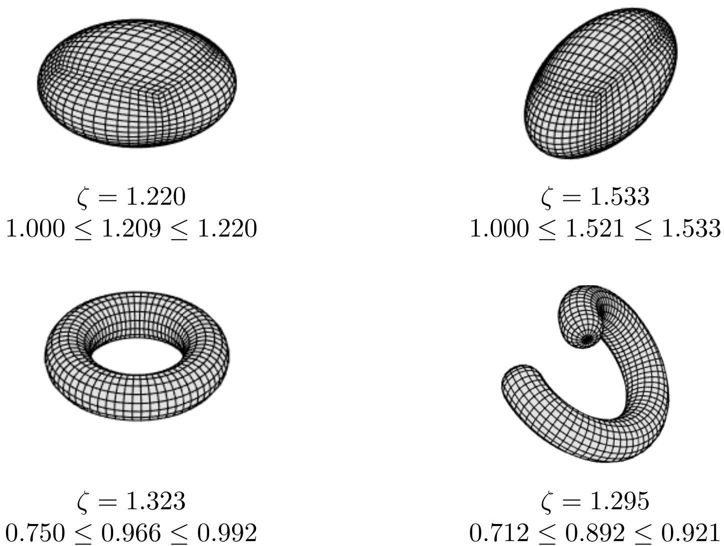

FIG. 1. *Numerically computed bounds for four example surfaces: oblate spheroid, prolate spheroid, torus, and helical tube. Shown for each surface is the bound ratio* ζ *and the triplet* $2\min_{x,y\in\Gamma}\frac{1}{|x-y|} \le 6\pi\mathcal{V} \le \mathrm{avg}_{x,y\in\Gamma}\frac{1}{|x-y|}$.

bound is computed from a discretization of the double surface integral using a Gaussian quadrature rule with a Duffy-type transformation [7] to treat the weakly singular integrand. All quantities are dimensionless as described earlier. The four surfaces we consider are an oblate spheroid with radii $(\rho_1,\rho_2,\rho_3) = (\frac{1}{2},1,1)$; a prolate spheroid with radii $(\rho_1,\rho_2,\rho_3) = (\frac{1}{2},\frac{1}{2},1)$; a torus with circular axial curve of radius $\rho = 1$ and circular cross sections of radius $r = \frac{1}{3}$; and a tube with helical axial curve of radius $\rho = 1$, pitch $\lambda = \frac{3}{2}$, arclength $\ell = 2\pi$, and circular cross sections of radius $r = \frac{1}{3}$, with hemispherical endcaps. Although all four surfaces have similar overall sizes, the examples show that ζ can vary significantly depending on the shape of the surface. Moreover, here we observe that the average velocity is closer to the upper bound rather than the lower, presumably due to the fact that the upper bound contains more specific information about the surface.

3. Proof. Here we provide a proof of Theorem 2.1. We begin with some necessary facts about the Stokes single-layer potentials and then outline a series of lemmas which lead to the main result. As in the previous section, we omit indices on vector and tensor quantities whenever there is no cause for confusion. Our most basic assumption will be that the surface Γ is closed and bounded and additionally is a Lyapunov surface [12], which briefly stated means that it has no self-intersections, is differentiable, and has a Hölder continuous outward unit normal field. Consistent with the assumptions on D^- and D^+, we assume that Γ is connected and hence consists of a single component.

3.1. Preliminaries. Let $\psi : \Gamma \to \mathbb{R}^3$ be a continuous function. Then by the Stokes single-layer velocity and pressure potentials on Γ with density ψ we mean

$$
\begin{aligned}
U_i[\Gamma,\psi](x) &= \int_\Gamma E^{ij}(x,y)\psi_j(y)\,dA_y,\\
P[\Gamma,\psi](x) &= \int_\Gamma \Pi^j(x,y)\psi_j(y)\,dA_y.
\end{aligned}
\tag{3.1}
$$

Here (E^{ij}, Π^j) is a fundamental solution of the Stokes equations referred to as a stokeslet; it is a solution of the free-space equations with a singular (Dirac) force at the point y [9, 28]. Using the notation $z = x - y$ and $r = |z|$, an explicit expression for this solution is

$$E^{ij}(x,y) = \frac{\delta_{ij}}{r} + \frac{z_i z_j}{r^3}, \quad \Pi^j(x,y) = \frac{2z_j}{r^3}. \tag{3.2}$$

We remark that, due to the linearity of the free-space equations, the above solution is defined up to an arbitrary choice of normalization. The choice of normalization naturally affects various constants in the developments that follow but is not crucial in any way; the choice adopted here is taken from [9].

For any continuous density ψ, the potentials $(U[\Gamma,\psi], P[\Gamma,\psi])$ are smooth at each $x \notin \Gamma$. Moreover, by virtue of their definitions as a linear combination of stokeslets, they satisfy the Stokes equations (2.1) at each $x \notin \Gamma$. While we consider the Stokes potentials with a density in the space of continuous functions, they could also be considered on various Sobolev spaces [17], but such generality will not be exploited here.

The velocity potential $U[\Gamma,\psi]$ is finite for all $x \in D^- \cup \Gamma \cup D^+$. In the special case when $x \in \Gamma$, the corresponding integral is only weakly singular and hence exists as an improper integral in the usual sense [12] provided that Γ is a Lyapunov surface. The restriction of $U[\psi,\Gamma]$ to Γ is denoted by $\overline{U}[\psi,\Gamma]$. This restriction is a continuous function on Γ; moreover, for any $x_0 \in \Gamma$, the following pointwise limit relations hold [24]:

$$\lim_{\substack{x \to x_0 \\ x \in D^+}} U[\Gamma,\psi](x) = \overline{U}[\Gamma,\psi](x_0), \tag{3.3}$$

$$\lim_{\substack{x \to x_0 \\ x \in D^-}} U[\Gamma,\psi](x) = \overline{U}[\Gamma,\psi](x_0). \tag{3.4}$$

Standard arguments [12] can be used to show that both of the above limits are approached uniformly in $x_0 \in \Gamma$. In contrast, the pressure potential $P[\Gamma,\psi]$ is not as simple. This potential is finite for all $x \in D^- \cup D^+$, but when $x \in \Gamma$, the corresponding integral is singular and hence does not exist as an improper integral in the usual sense. Nevertheless, when the density ψ and surface Γ are sufficiently regular, the potential $P[\Gamma,\psi]$ has pointwise limits as x approaches Γ [24, 27, 30]. Notice that the existence of such limits is intimately connected with regularity up to the boundary of solutions of the Stokes equations.

The single-layer stress potential associated with $(U[\Gamma,\psi], P[\Gamma,\psi])$ is

$$\Sigma^{ik}[\Gamma,\psi](x) = \int_\Gamma \Xi^{ikj}(x,y)\psi_j(y)\, dA_y, \tag{3.5}$$

where Ξ^{ikj} is the stress field corresponding to the stokeslet solution in (3.2). In particular, we have [9, 28]

$$\Xi^{ikj}(x,y) = -\frac{6z_i z_k z_j}{r^5}. \tag{3.6}$$

The limiting traction field on Γ associated with $\Sigma[\Gamma,\psi]$ is also of interest. To characterize it, we consider a neighborhood of Γ, with points parameterized as $x_\tau = x_0 + \tau\nu(x_0)$, where $x_0 \in \Gamma$ and $\tau \in [-\epsilon,\epsilon]$ for some $\epsilon > 0$ sufficiently small, and we extend the normal field such that $\nu(x_\tau) = \nu(x_0)$. In this neighborhood, we have the traction potential

$$H^i[\Gamma,\psi](x_\tau) = \Sigma^{ik}[\Gamma,\psi](x_\tau)\nu_k(x_\tau). \tag{3.7}$$

For any continuous density ψ, the stress potential $\Sigma[\Gamma,\psi]$ is smooth at each $x \notin \Gamma$. At any such point, this potential provides an explicit expression for the stress associated with the velocity-pressure pair $(U[\Gamma,\psi], P[\Gamma,\psi])$. Moreover, for any fixed $\tau \neq 0$, the set of points x_τ define a surface Γ_τ that is parallel to Γ and has the same normal field, and the traction potential $H[\Gamma,\psi]$ provides an explicit expression for the associated traction on this surface.

The traction potential $H[\Gamma,\psi]$ is finite for all $x_0 \in \Gamma$ and $\tau \in [-\epsilon,\epsilon]$. When $\tau = 0$, the corresponding integral is only weakly singular and hence exists as an improper integral provided that Γ is a Lyapunov surface. The restriction of $H[\Gamma,\psi]$ to Γ ($\tau = 0$) is denoted by $\overline{H}[\Gamma,\psi]$. This restriction is a continuous function on Γ; moreover, for any $x_0 \in \Gamma$, the following pointwise limit relations hold [24, 27]:

$$\lim_{\tau\to 0^+} H[\Gamma,\psi](x_\tau) = -4\pi\psi(x_0) + \overline{H}[\Gamma,\psi](x_0), \tag{3.8}$$

$$\lim_{\tau\to 0^-} H[\Gamma,\psi](x_\tau) = \quad 4\pi\psi(x_0) + \overline{H}[\Gamma,\psi](x_0). \tag{3.9}$$

Similar to before, standard arguments [12] can be used to show that both of the above limits are approached uniformly in $x_0 \in \Gamma$. Notice that, by continuity of ψ and $\overline{H}[\Gamma,\psi]$, the one-sided limits in (3.8) and (3.9) are themselves continuous functions on Γ.

3.2. Lemmata. We begin by summarizing a classic result on the solvability of the exterior Stokes system. We omit the proof but note that the result follows from the Fredholm theorems for integral equations with compact operators by considering appropriate potential representations of the fields (u^+, p^+); for details, see [9, 17, 24, 27].

LEMMA 3.1. *Let Γ be a closed, bounded Lyapunov surface. Then the exterior Stokes system* (2.1)–(2.3) *has a unique solution (u^+, p^+) for any continuous data v. The fields (u^+, p^+) are smooth at each point in the open domain D^+, and the velocity u^+ is continuous up to the boundary Γ.*

The next result that we summarize is also classic. It states that for rigid-body boundary data, the unique solution of the exterior Stokes system can be represented with only single-layer potentials. In this case, the density in the representation is proportional to the traction field on the surface, up to a multiple of the unit normal field. Moreover, when the unit normal field is extended to a neighborhood of the surface by being constant along normal lines, the traction field in this neighborhood has the structure of a double-layer potential and is well-behaved up to the surface as outlined in (3.5)–(3.9). For brevity we omit the proofs; for details, see [17, 24, 28].

LEMMA 3.2. *In addition to the conditions of Lemma* 3.1, *assume that the data v is of the rigid-body form* (2.4) *with a given reference point c. Then the Stokes solution (u^+, p^+) can be represented by single-layer potentials with a continuous density ψ, namely,*

$$u^+(x) = U[\Gamma,\psi](x), \quad p^+(x) = P[\Gamma,\psi](x), \quad x \in D^+. \tag{3.10}$$

The density ψ in this representation is unique up to an element of $\operatorname{span}\{\nu\}$, *which corresponds to the nullspace of the single-layer velocity potential restricted to Γ. Moreover, the traction $\sigma^+[u^+,p^+]\nu$ is continuous up to the boundary Γ and we have*

$$\int_\Gamma \sigma^+[u^+,p^+](x)\nu(x)\, dA_x = -8\pi \int_\Gamma \psi(x)\, dA_x, \tag{3.11}$$

$$\int_\Gamma (x-c)\times\sigma^+[u^+,p^+](x)\nu(x)\, dA_x = -8\pi \int_\Gamma (x-c)\times\psi(x)\, dA_x. \tag{3.12}$$

The next result establishes some important properties of the restriction of the single-layer velocity potential to the surface $\mathit{\Gamma}$. In addition to being symmetric, the fact that the potential is semi-positive-definite and strictly positive up to a one-dimensional subspace will play a key role in our developments. We state the result only for continuous vector fields and note that stronger results, for example, coercivity up to a one-dimensional subspace, can be established in more general function spaces [6, 17]. Here and throughout we use $\mathcal{C}(\mathit{\Gamma})$ to denote the space of continuous vector fields on $\mathit{\Gamma}$. Also, for any square matrix Q we use the notation $Q : Q = Q_{ij}Q_{ij} = |Q|^2$ and $\mathrm{sym}(Q) = (Q + Q^T)/2$.

LEMMA 3.3. *Let $\mathit{\Gamma}$ be a closed, bounded Lyapunov surface. Then the single-layer velocity potential $\overline{U}[\mathit{\Gamma}, \psi]$ on $\mathit{\Gamma}$ is symmetric and semi-positive-definite on the space $\mathcal{C}(\mathit{\Gamma})$ in the sense that*

$$\int_{\mathit{\Gamma}} \varphi(x) \cdot \overline{U}[\mathit{\Gamma}, \psi](x)\, dA_x = \int_{\mathit{\Gamma}} \psi(x) \cdot \overline{U}[\mathit{\Gamma}, \varphi](x)\, dA_x \quad \forall \psi, \varphi \in \mathcal{C}(\mathit{\Gamma}), \tag{3.13}$$

$$\int_{\mathit{\Gamma}} \psi(x) \cdot \overline{U}[\mathit{\Gamma}, \psi](x)\, dA_x \geq 0 \quad \forall \psi \in \mathcal{C}(\mathit{\Gamma}), \tag{3.14}$$

$$\int_{\mathit{\Gamma}} \psi(x) \cdot \overline{U}[\mathit{\Gamma}, \psi](x)\, dA_x = 0 \quad \text{iff} \quad \overline{U}[\mathit{\Gamma}, \psi] \equiv 0 \quad \text{iff} \quad \psi \in \mathrm{span}\{\nu\}. \tag{3.15}$$

Proof. Symmetry follows directly from (3.1) and (3.2) and the fact that $E^{ij}(x, y)$ is symmetric in the indices i, j and the points x, y. To establish semipositivity, let $\psi \in \mathcal{C}(\mathit{\Gamma})$ be arbitrary, and let (u^-, p^-) and (u^+, p^+) be the Stokes fields in D^- and D^+ generated by the single-layer velocity and pressure potentials on $\mathit{\Gamma}$ with density ψ. The pair (u^-, p^-) has an associated stress field σ^- in D^- and traction field h^- on $\mathit{\Gamma}$, and similarly (u^+, p^+) has an associated stress field σ^+ in D^+ and traction field h^+ on $\mathit{\Gamma}$. The tractions h^- and h^+ are both defined using the outward unit normal field ν on $\mathit{\Gamma}$. In view of the limit relations in (3.3)–(3.4) and (3.8)–(3.9) we have, for any $x_0 \in \mathit{\Gamma}$,

$$u^+(x_0) = \overline{U}[\mathit{\Gamma}, \psi](x_0), \quad u^-(x_0) = \overline{U}[\mathit{\Gamma}, \psi](x_0), \tag{3.16}$$

$$h^+(x_0) = -4\pi\psi(x_0) + \overline{H}[\mathit{\Gamma}, \psi](x_0), \quad h^-(x_0) = 4\pi\psi(x_0) + \overline{H}[\mathit{\Gamma}, \psi](x_0), \tag{3.17}$$

from which we deduce that $8\pi\psi(x_0) = h^-(x_0) - h^+(x_0)$. Using these observations, together with the divergence theorem in each of D^- and D^+, and the fact that (u^-, p^-) and (u^+, p^+) satisfy (2.1) in the domains D^- and D^+, with (u^+, p^+) satisfying the decay conditions (2.3) at infinity, and the fact that ν points into D^+, we get

$$\begin{aligned}
\int_{\mathit{\Gamma}} \psi \cdot \overline{U}[\mathit{\Gamma}, \psi]\, dA_x &= \frac{1}{8\pi} \int_{\mathit{\Gamma}} [h_i^- - h_i^+] \overline{U}_i[\mathit{\Gamma}, \psi]\, dA_x \\
&= \frac{1}{8\pi} \int_{\mathit{\Gamma}} h_i^- u_i^-\, dA_x - \frac{1}{8\pi} \int_{\mathit{\Gamma}} h_i^+ u_i^+\, dA_x \\
&= \frac{1}{8\pi} \int_{\mathit{\Gamma}} u_i^- \sigma_{ij}^- \nu_j\, dA_x - \frac{1}{8\pi} \int_{\mathit{\Gamma}} u_i^+ \sigma_{ij}^+ \nu_j\, dA_x \\
&= \frac{1}{8\pi} \int_{D^-} (u_i^- \sigma_{ij}^-)_{,j}\, dV_x + \frac{1}{8\pi} \int_{D^+} (u_i^+ \sigma_{ij}^+)_{,j}\, dV_x \\
&= \frac{1}{4\pi} \int_{D^-} |\,\mathrm{sym}(\nabla u^-)|^2\, dV_x + \frac{1}{4\pi} \int_{D^+} |\,\mathrm{sym}(\nabla u^+)|^2\, dV_x.
\end{aligned} \tag{3.18}$$

Hence $\int_{\mathit{\Gamma}} \psi \cdot \overline{U}[\mathit{\Gamma}, \psi]\, dA_x \geq 0$ for arbitrary $\psi \in \mathcal{C}(\mathit{\Gamma})$. Moreover, $\int_{\mathit{\Gamma}} \psi \cdot \overline{U}[\mathit{\Gamma}, \psi]\, dA_x = 0$ implies that $\mathrm{sym}(\nabla u^+) \equiv 0$ in D^+, which implies that u^+ is a rigid-body velocity field in D^+. Moreover, since $u^+ \to 0$ as $|x| \to \infty$, we deduce that $u^+ \equiv 0$ in D^+ and also on Γ. Thus $\overline{U}[\Gamma,\psi] \equiv 0$ on Γ and hence ψ is in the nullspace, which is known to be span$\{\nu\}$. Conversely, if ψ is in span$\{\nu\}$, then $\overline{U}[\Gamma,\psi] \equiv 0$ on Γ and hence $\int_\Gamma \psi \cdot \overline{U}[\Gamma,\psi]\, dA_x = 0$. ◻

The next result establishes some important properties of the matrices L and M associated with the exterior Stokes system when the boundary data is restricted to the six-dimensional space of rigid-body vector fields on Γ. The symmetry and positive-definite properties of L, and consequently also of $M = L^{-1}$, are closely related to the properties outlined above for the single-layer velocity potential.

Lemma 3.4. *Let Γ be a closed, bounded Lyapunov surface, and let c be an arbitrary reference point. Then the resistance matrix $L \in \mathbb{R}^{6\times 6}$ introduced in* (2.9) *is symmetric and positive-definite. Specifically, for any $V \in \mathbb{R}^3$ and $\Omega \in \mathbb{R}^3$ such that $(V,\Omega) \neq (0,0)$, we have*

$$(V,\Omega)\cdot L(V,\Omega) = \int_{D^+} 2|\operatorname{sym}(\nabla u^+(x))|^2\, dV_x > 0, \tag{3.19}$$

where (u^+,p^+) is the unique solution of the exterior Stokes system (2.1)–(2.3) *with data of the rigid-body form* (2.4).

Proof. To establish symmetry, let two arbitrary sets of rigid-body velocities (V,Ω) and $(\widehat{V},\widehat{\Omega})$ be given, with the same reference point c. Let (u^+,p^+) and $(\widehat{u}^+,\widehat{p}^+)$ be the corresponding solutions of the exterior Stokes system in D^+. The pair (u^+,p^+) has an associated stress field σ^+, and traction field h^+ and load resultants (F,T) on Γ. Similarly, the pair $(\widehat{u}^+,\widehat{p}^+)$ has an associated stress field $\widehat{\sigma}^+$, and traction field $\widehat{h}^+$ and load resultants $(\widehat{F},\widehat{T})$ on Γ. By Lemma 3.2, the pairs (u^+,p^+) and $(\widehat{u}^+,\widehat{p}^+)$ can be represented by single-layer potentials with densities ψ and $\widehat{\psi}$ on Γ. Moreover, by definition of L, we have $(F,T) = -L(V,\Omega)$ and $(\widehat{F},\widehat{T}) = -L(\widehat{V},\widehat{\Omega})$. Making direct use of (3.11) and (3.12) from Lemma 3.2, along with (2.8), (2.4), and $(2.2)_1$, we get

$$\begin{aligned}
(\widehat{V},\widehat{\Omega})\cdot L(V,\Omega) &= -\widehat{V}\cdot F - \widehat{\Omega}\cdot T \\
&= 8\pi \int_\Gamma \widehat{V}\cdot\psi + \widehat{\Omega}\cdot((x-c)\times\psi)\, dA_x \\
&= 8\pi \int_\Gamma \widehat{u}^+\cdot\psi\, dA_x \\
&= 8\pi \int_\Gamma \psi\cdot\overline{U}[\Gamma,\widehat{\psi}]\, dA_x.
\end{aligned} \tag{3.20}$$

The symmetry of L then follows from the symmetry of the single-layer velocity potential established in Lemma 3.3.

To establish positivity, we consider (3.20) with $(\widehat{V},\widehat{\Omega}) = (V,\Omega)$ to obtain

$$(V,\Omega)\cdot L(V,\Omega) = 8\pi \int_\Gamma \psi\cdot\overline{U}[\Gamma,\psi]\, dA_x. \tag{3.21}$$

From Lemma 3.3 we then deduce that $(V,\Omega)\cdot L(V,\Omega) \geq 0$, with equality occurring only when $\psi \in \operatorname{span}\{\nu\}$, which implies that on Γ we have $u^+ = \overline{U}[\Gamma,\psi] \equiv 0$ and hence that $(V,\Omega) = (0,0)$. To establish the final result we note from (3.21) and (3.18) that

$$(V,\Omega)\cdot L(V,\Omega) = \int_{D^-} 2|\operatorname{sym}(\nabla u^-)|^2\, dV_x + \int_{D^+} 2|\operatorname{sym}(\nabla u^+)|^2\, dV_x. \tag{3.22}$$

By uniqueness for the interior Stokes system, the field u^- in D^- is a rigid-body velocity field determined by the boundary data, and hence $\mathrm{sym}(\nabla u^-) \equiv 0$ in D^-, which establishes the result in (3.19). ∎

The next result establishes the unique solvability of a Neumann-type problem for the exterior Stokes system; it is often referred to as a rigid-body mobility problem [22]. We omit the proof, but note that it is a straightforward consequence of the definition and invertibility of the matrices L and M. Here and throughout we use $\mathcal{R}(\Gamma)$ to denote the six-dimensional space of rigid-body vector fields on Γ.

LEMMA 3.5. *Let Γ be a closed, bounded Lyapunov surface and let c be an arbitrary reference point. Then for any given $F \in \mathbb{R}^3$ and $T \in \mathbb{R}^3$ the Stokes system*

$$\begin{aligned} \Delta u^+ &= \nabla p^+, && x \in D^+, \\ \nabla \cdot u^+ &= 0, && x \in D^+, \\ u^+, p^+ &\to 0, && |x| \to \infty, \end{aligned} \tag{3.23}$$

$$u^+|_\Gamma \in \mathcal{R}(\Gamma), \quad \int_\Gamma \sigma^+[u^+,p^+]\nu \, dA_x = F, \quad \int_\Gamma (x-c) \times \sigma^+[u^+,p^+]\nu \, dA_x = T \tag{3.24}$$

has a unique solution (u^+, p^+). Specifically, there are unique $V \in \mathbb{R}^3$ and $\Omega \in \mathbb{R}^3$ such that

$$u^+(x) = V + \Omega \times (x-c), \qquad x \in \Gamma. \tag{3.25}$$

The following result shows that the flow field (u^+, p^+) in Lemma 3.5 minimizes the so-called dissipation integral among all exterior flow fields (v^+, q^+) with the same resultant force and torque about the same reference point c, which can be interpreted as a weak type of Neumann data. Various such extremum principles have been described for Stokes flow [16, 18, 19, 21, 29]. Here we outline one such principle in a form that is convenient for our purposes. This result will be crucial in establishing the lower bound in Theorem 2.1.

LEMMA 3.6. *Let Γ be a closed, bounded Lyapunov surface and let c be an arbitrary reference point. For any given $F \in \mathbb{R}^3$ and $T \in \mathbb{R}^3$, let (u^+, p^+) be the unique solution of (3.23)–(3.24), and let (v^+, q^+) be any smooth fields in D^+, with v^+ and $\sigma^+[v^+, q^+]\nu$ continuous up to Γ, satisfying*

$$\begin{aligned} \Delta v^+ &= \nabla q^+, && x \in D^+, \\ \nabla \cdot v^+ &= 0, && x \in D^+, \\ v^+, q^+ &\to 0, && |x| \to \infty, \end{aligned} \tag{3.26}$$

$$v^+|_\Gamma \in \mathcal{C}(\Gamma), \quad \int_\Gamma \sigma^+[v^+,q^+]\nu \, dA_x = F, \quad \int_\Gamma (x-c) \times \sigma^+[v^+,q^+]\nu \, dA_x = T. \tag{3.27}$$

Then

$$\int_{D^+} |\,\mathrm{sym}(\nabla u^+(x))|^2 \, dV_x \le \int_{D^+} |\,\mathrm{sym}(\nabla v^+(x))|^2 \, dV_x. \tag{3.28}$$

Proof. Let $w^+ = v^+ - u^+$ and $\eta^+ = q^+ - p^+$. Then by employing the conditions in (3.23)–(3.24) and (3.26)–(3.27), together with the divergence theorem and the fact that ν points into D^+, and the result in (3.25), we obtain

$$
\begin{aligned}
&\int_{D^+} 2\,\mathrm{sym}(\nabla w^+) : \mathrm{sym}(\nabla u^+)\; dV_x \\
&\quad = \int_{D^+} \sigma^+[w^+,\eta^+] : \nabla u^+\; dV_x \\
&\quad = -\int_{\Gamma} u^+ \cdot \sigma^+[w^+,\eta^+]\nu\; dA_x \\
&\quad = -\int_{\Gamma} (V + \Omega \times (x-c)) \cdot \sigma^+[w^+,\eta^+]\nu\; dA_x = 0,
\end{aligned} \tag{3.29}
$$

where the last equality follows from the fact that the traction $\sigma^+[w^+,\eta^+]\nu$ has zero resultant force and torque on Γ when the same reference point c is chosen in (3.24) and (3.27), that is, $\int_\Gamma \sigma^+[w^+,\eta^+]\nu\; dA_x = 0$ and $\int_\Gamma (x-c) \times \sigma^+[w^+,\eta^+]\nu\; dA_x = 0$. Using the orthogonality condition in (3.29) we find

$$
\begin{aligned}
&\int_{D^+} |\,\mathrm{sym}(\nabla v^+)|^2 - |\,\mathrm{sym}(\nabla u^+)|^2\; dV_x \\
&\quad = \int_{D^+} \mathrm{sym}(\nabla w^+) : \mathrm{sym}(\nabla v^+ + \nabla u^+)\; dV_x \\
&\quad = \int_{D^+} \mathrm{sym}(\nabla w^+) : \mathrm{sym}(\nabla v^+ - \nabla u^+)\; dV_x \\
&\quad = \int_{D^+} |\,\mathrm{sym}(\nabla w^+)|^2\; dV_x \geq 0,
\end{aligned} \tag{3.30}
$$

which establishes the desired result. ◻

The following result shows that the density ψ in the single-layer representation of the flow field (u^+, p^+) of Lemma 3.5 also satisfies an extremum principle. For given loads (F,T), we note that the associated velocities (V,Ω) arise as Lagrange multipliers. This constrained extremum principle appears to be not well-known and will play a key role in establishing the upper bound in Theorem 2.1. To state the result, we consider two linear functionals on $\mathcal{C}(\Gamma)$ defined by

$$
\mathcal{F}[\psi] = -8\pi \int_\Gamma \psi(x)\; dA_x, \quad \mathcal{T}[\psi] = -8\pi \int_\Gamma (x-c) \times \psi(x)\; dA_x. \tag{3.31}
$$

LEMMA 3.7. *Let Γ be a closed, bounded Lyapunov surface and let c be an arbitrary reference point. For any given loads $F \in \mathbb{R}^3$ and $T \in \mathbb{R}^3$, let (u^+, p^+) be the unique solution of (3.23)–(3.24). Then u^+ and p^+ have the single-layer representation*

$$
u^+(x) = U[\Gamma,\psi_*](x), \quad p^+(x) = P[\Gamma,\psi_*](x), \quad x \in D^+, \tag{3.32}
$$

where ψ_* is a continuous density satisfying

$$
\psi_* = \underset{\substack{\psi \in \mathcal{C}(\Gamma) \\ \mathcal{F}[\psi]=F,\, \mathcal{T}[\psi]=T}}{\operatorname{argmin}} \left[\int_\Gamma \psi(x) \cdot \overline{U}[\Gamma,\psi](x)\; dA_x \right]. \tag{3.33}
$$

The optimality conditions for the density ψ_* are

$$
\overline{U}[\Gamma,\psi_*](x) = V + \Omega \times (x-c), \quad x \in \Gamma, \tag{3.34}
$$

where $V \in \mathbb{R}^3$ and $\Omega \in \mathbb{R}^3$ are the rigid-body velocities associated with (u^+, p^+).

Proof. Given (F, T) and c, let (u^+, p^+) be the unique solution of (3.23)–(3.24). In view of (3.25) and Lemma 3.2, the fields u^+ and p^+ have the single-layer representation (3.32) for some continuous ψ_*. Moreover, combining (3.25) with (3.3), we deduce that ψ_* must satisfy (3.34).

We next show directly that ψ_* must be a minimizer of (3.33). First, we note that, by (3.24) and Lemma 3.2, we have $\mathcal{F}[\psi_*] = F$ and $\mathcal{T}[\psi_*] = T$. Next, for any $\psi \in \mathcal{C}(\Gamma)$ such that $\mathcal{F}[\psi] = F$ and $\mathcal{T}[\psi] = T$, we note that

$$\int_\Gamma (\psi - \psi_*) \cdot \overline{U}[\Gamma, \psi_*]\, dA_x = \int_\Gamma (\psi - \psi_*) \cdot (V + \Omega \times (x - c))\, dA_x = 0, \tag{3.35}$$

where the last equality follows from the fact that $\mathcal{F}[\psi - \psi_*] = 0$ and $\mathcal{T}[\psi - \psi_*] = 0$. Using the orthogonality condition in (3.35), and the symmetry and semipositivity properties from Lemma 3.3, we find

$$\begin{aligned}\int_\Gamma \psi \cdot \overline{U}[\Gamma, \psi] - \psi_* \cdot \overline{U}[\Gamma, \psi_*]\, dA_x &= \int_\Gamma (\psi - \psi_*) \cdot \overline{U}[\Gamma, \psi + \psi_*]\, dA_x \\ &= \int_\Gamma (\psi - \psi_*) \cdot \overline{U}[\Gamma, \psi - \psi_*]\, dA_x \geq 0,\end{aligned} \tag{3.36}$$

which shows that ψ_* is a minimizer. Moreover, we note that the conditions in (3.34) can be interpreted as the first-order necessary conditions for the minimization in (3.33), with V and Ω playing the roles of multipliers for the linear constraints $\mathcal{F}[\psi] = F$ and $\mathcal{T}[\psi] = T$. Finally, by Lemma 3.2, we remark that minimizers are only determined to within an element of span$\{\nu\}$. ∎

3.3. Proof of Theorem 2.1. The main result can now be proved. We consider the upper and lower bounds separately.

Upper bound. For given (F, T) and c, let (u^+, p^+) be the unique solution of (3.23)–(3.24) with associated velocities (V, Ω), so that $(V, \Omega) = -M(F, T)$ and $(F, T) = -L(V, \Omega)$, and let ψ_* be a density as in Lemma 3.7. Using the facts that $\mathcal{F}[\psi_*] = F$ and $\mathcal{T}[\psi_*] = T$, together with (3.34), we find

$$\begin{aligned}(F, T) \cdot M(F, T) &= -F \cdot V - T \cdot \Omega \\ &= -\mathcal{F}[\psi_*] \cdot V - \mathcal{T}[\psi_*] \cdot \Omega \\ &= 8\pi \int_\Gamma \psi_* \cdot V\, dA_x + 8\pi \int_\Gamma \psi_* \cdot (\Omega \times (x - c))\, dA_x \\ &= 8\pi \int_\Gamma \psi_* \cdot \overline{U}[\Gamma, \psi_*]\, dA_x,\end{aligned} \tag{3.37}$$

which by the extremum properties of ψ_* established in Lemma 3.7 implies

$$(F, T) \cdot M(F, T) \leq 8\pi \int_\Gamma \psi \cdot \overline{U}[\Gamma, \psi]\, dA_x \quad \forall \psi \in \mathcal{C}(\Gamma) :\ \mathcal{F}[\psi] = F,\ \mathcal{T}[\psi] = T. \tag{3.38}$$

An explicit upper bound can be obtained by making explicit choices for ψ and c and using the integral expression for $\overline{U}[\Gamma, \psi]$. Specifically, let c be the centroid of Γ, let $F = n$ be a unit vector, and let $T = 0$. Then the constant density $\psi(x) \equiv -n/(8\pi|\Gamma|)$, where $|\Gamma|$ denotes the area of Γ, satisfies $\mathcal{F}[\psi] = n$ and $\mathcal{T}[\psi] = 0$, and from (3.38) we obtain

$$(n, 0) \cdot M(n, 0) \leq \frac{1}{8\pi|\Gamma|^2} \int_\Gamma n \cdot \overline{U}[\Gamma, n]\, dA_x. \tag{3.39}$$

Using the block structure of M in (2.11) and the explicit expression for the single-layer potential $\overline{U}$ obtained from (3.1) and (3.2), we get

$$n \cdot M_1 n \leq \frac{1}{8\pi|\Gamma|^2} \int_\Gamma \int_\Gamma \left[\frac{1}{r} + \frac{(z\cdot n)^2}{r^3}\right] \, dA_y \, dA_x, \tag{3.40}$$

where $z = x - y$ and $r = |z|$. Replacing n by each of the standard unit vectors e_i, and summing the results, using the fact that $|z|^2 = \sum_i (z \cdot e_i)^2$, we obtain

$$\frac{1}{3}\operatorname{tr}(M_1) \leq \frac{1}{6\pi|\Gamma|^2} \int_\Gamma \int_\Gamma \frac{1}{r} \, dA_y \, dA_x, \tag{3.41}$$

which in view of (2.15) establishes the upper bound in Theorem 2.1.

We remark that if c was chosen to be some point other than the centroid, then the conditions $\mathcal{F}[\psi] = n$ and $\mathcal{T}[\psi] = 0$ would no longer be satisfied with a constant density as above; $\mathcal{T}[\psi] = 0$ would not hold. In this case, we could consider a more general density of the linear form $\psi(x) = a + b \times (x - c)$, where a and b are constant vectors that can be uniquely chosen to satisfy the conditions $\mathcal{F}[\psi] = F$ and $\mathcal{T}[\psi] = T$ for any F and T. Such a density could then be used as above to obtain an upper bound, but the resulting expression is rather complicated and its physical interpretation is less straightforward. Sharper bounds could be obtained by considering more general choices of ψ, and optimizing over free parameters, but at the expense of simplicity and ease of interpretation of the results.

Lower bound. Consider a fictitious second body domain $\widehat{D}^-$ with associated exterior domain $\widehat{D}^+$ and bounding surface $\widehat{\Gamma}$. We assume that $D^- \subset \widehat{D}^-$ so that Γ is enclosed by $\widehat{\Gamma}$, and moreover $\widehat{D}^+ \subset D^+$. Furthermore, let c and $\widehat{c}$ be reference points for Γ and $\widehat{\Gamma}$. We will require that $\widehat{c} = c$ and assume as above that c is the centroid of Γ. However, we note that the results below actually hold for arbitrary choices of c.

For given (F, T) and c, let (u^+, p^+) be the unique solution of (3.23)–(3.24) with associated velocities (V, Ω), so that $(V, \Omega) = -M(F, T)$ and $(F, T) = -L(V, \Omega)$. Also, for the same given (F, T) and for $\widehat{c} = c$, let $(\widehat{u}^+, \widehat{p}^+)$ be the unique solution of (3.23)–(3.24) for the surface $\widehat{\Gamma}$, with associated velocities $(\widehat{V}, \widehat{\Omega})$ satisfying $(\widehat{V}, \widehat{\Omega}) = -\widehat{M}(F, T)$ and $(F, T) = -\widehat{L}(\widehat{V}, \widehat{\Omega})$, where $\widehat{L}$ and $\widehat{M}$ are the resistance and mobility matrices associated with $\widehat{\Gamma}$. Moreover, notice that (u^+, p^+) are fields satisfying (3.26)–(3.27) for the surface $\widehat{\Gamma}$ and domain $\widehat{D}^+$. Then by Lemma 3.4, the inclusion $\widehat{D}^+ \subset D^+$, and Lemma 3.6, we have

$$\begin{aligned}
(F, T) \cdot M(F, T) &= (V, \Omega) \cdot L(V, \Omega) \\
&= \int_{D^+} 2|\operatorname{sym}(\nabla u^+)|^2 \, dV_x \\
&\geq \int_{\widehat{D}^+} 2|\operatorname{sym}(\nabla u^+)|^2 \, dV_x \\
&\geq \int_{\widehat{D}^+} 2|\operatorname{sym}(\nabla \widehat{u}^+)|^2 \, dV_x \\
&= (\widehat{V}, \widehat{\Omega}) \cdot \widehat{L}(\widehat{V}, \widehat{\Omega}) = (F, T) \cdot \widehat{M}(F, T).
\end{aligned} \tag{3.42}$$

An explicit lower bound can be obtained by considering an enclosing surface $\widehat{\Gamma}$ for which the matrix $\widehat{M}$ is known explicitly. Specifically, let $\widehat{\Gamma}$ be a sphere of diameter $\widehat{d}$, where $\widehat{d} = \max_{x,y\in\Gamma} |x - y|$ is also the diameter of Γ, and let $\widehat{c}^0$ be the center of the sphere, and notice that in general $\widehat{c}^0 \neq \widehat{c}$. Using classic results for Stokes flow around a sphere [13, 22], the mobility matrix $\widehat{M}^0$ with respect to $\widehat{c}^0$ is diagonal, with diagonal blocks $\widehat{M}_1^0 = (3\pi\widehat{d})^{-1}I$ and $\widehat{M}_4^0 = (\pi\widehat{d}^3)^{-1}I$, where I denotes the 3×3 identity matrix. Moreover, by well-known translation formulae for the mobility matrix [11, 14], the corresponding matrix $\widehat{M}$ with respect to the point $\widehat{c}$ can be obtained, and focusing on the submatrix $\widehat{M}_1$ we have

$$\widehat{M}_1 = (3\pi\widehat{d})^{-1}I + (\pi\widehat{d}^3)^{-1}[|\delta|^2 I - \delta \otimes \delta], \tag{3.43}$$

where $\delta = \widehat{c} - \widehat{c}^0$ and $\otimes$ denotes an outer product. Considering (3.42), setting $T = 0$, and taking F to be each of the standard unit vectors e_i, and summing results, leads to

$$\frac{1}{3}\operatorname{tr}(M_1) \geq \frac{1}{3\pi\widehat{d}} + \frac{2|\delta|^2}{3\pi\widehat{d}^3} \geq \frac{1}{3\pi\widehat{d}} = \frac{1}{3\pi \max_{x,y\in\Gamma}|x-y|}, \tag{3.44}$$

which in view of (2.15) establishes the lower bound in Theorem 2.1. We remark that other bounds could be obtained by making different choices for the enclosing surface $\widehat{\Gamma}$, for instance, an ellipsoid, but at the expense of simplicity and explicitness. ∎

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