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A THEOREM ON THE SURFACE TRACTION FIELD IN POTENTIAL REPRESENTATIONS OF STOKES FLOW*

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Abstract. A characterization theorem for moments of the Stokes traction field on the bounding surface of a three-dimensional flow domain is stated and proved. Whereas the single-layer Stokes potentials lead to a weakly singular representation of the traction field on the boundary, the double-layer potentials lead to a hyper-singular representation, which is less convenient for analysis and numerics. However, in various applications, pointwise values of the traction field up to the boundary are not of direct interest, but rather moments of the traction with respect to given weighting functions. It is shown that such moments can be characterized in terms of weakly singular integrals for both the single- and double-layer potentials, and moreover, the characterization does not involve derivatives of the potential densities or the moment weighting functions. The result for the double-layer potentials can be viewed as a generalization and enhancement of the classic Lyapunov–Tauber theorem and may make these potentials easier to use. An example application to the modeling of immersed flexible bodies is discussed.

Key words. Stokes equations, layer potentials, weakly singular integrals, hyper-singular integrals, Lyapunov–Tauber theorem

AMS subject classifications. 31B10, 35Q30, 76D07, 74F10

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1. Introduction. The hydromechanics of slow viscous flow modeled by the Stokes equations plays an important role in many different areas of science and engineering. Classic applications range from the study of the rheology of colloidal suspensions [9, 18] and lubrication theory [33], to the study of the locomotion and swimming of microorganisms [22]. Modern applications range from the study of structural properties of macromolecules such as proteins and DNA in biophysical chemistry [2], to the modeling of various devices for the separation and manipulation of particles in micro-fluidic systems [19]. In such applications, the Stokes equations provide a first approximation of the more general Navier–Stokes equations in regimes where the flow is nearly steady and slow, with small velocity gradients. The Stokes equations can be used to approximate a variety of different flow quantities, ranging from velocity streamlines and pressure gradients to drag forces and torques on immersed bodies, and more general fluid-structure interactions.

Different approaches are available for the formulation and study of Stokes flow problems in three dimensions. One approach is to consider a boundary value problem in the form of partial differential equations for the fluid velocity and pressure fields in a domain of interest [20]. Another approach is to reduce the boundary value problem to an integral equation on the bounding surface of the domain [11, 29, 31]. The unknown velocity and pressure fields throughout the domain are represented in terms of one or more integral potentials which depend on an unknown surface density. The integral potentials are usually defined in terms of the classic point-force and point-source fundamental solutions of the Stokes equations and combinations of derivatives thereof, referred to as stokeslets, stresslets, and rotlets [29, 31]. Moreover, in variants

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of the standard approach, the unknown density in the integral potential need not be supported on the bounding surface of the domain, but instead may be supported on a related curve as considered in the slender-body method [1, 13, 17], or supported on an arbitrary set of points as considered in more general singularity methods [3, 4, 5, 31].

Here we study properties of the classic single- and double-layer Stokes potentials. These potentials are defined in terms of the fundamental stokeslet and stresslet solutions, and together they form the basis for a variety of boundary integral methods for the analytical [11, 20, 26] and numerical [7, 10, 28, 29, 31] treatment of the Stokes equations. We are specifically interested in the traction field on the boundary, which corresponds to the fluid force per unit area, generated by these potentials. Whereas the single-layer potentials lead to a weakly singular representation of the traction field on the boundary, the double-layer potentials lead to a hyper-singular representation, which is less convenient for analysis and numerics. However, rather than focus attention on pointwise values of the traction field up to the boundary, which indeed may or may not exist at all points depending on the regularity of the boundary surface and data, we instead focus attention on moments of the traction field with respect to arbitrary weighting functions. Such moments arise naturally in a variety of contexts, for example, the determination of drag forces and torques associated with the translational and rotational dynamics, and more general hydrodynamic loads associated with the shape dynamics, of rigid and flexible bodies.

We show that general traction moments, for both the single- and double-layer potentials, can be characterized in terms of weakly singular integrals depending only pointwise on the potential density and moment weight functions. The results for the double-layer case are obtained without any explicit application of Stokes theorem or integration by parts, as have been considered in other contexts [11, 25], which would generally lead to expressions involving surface derivatives of the density and weight. Our results in this case can be viewed as a generalization and enhancement of the Lyapunov–Tauber theorem of classic potential theory [8] and its analogue for the Stokes equations [29]. This theorem asserts that the one-sided pointwise limits of the double-layer traction field as the boundary is approached from either side either exist and are equal or else do not exist; sufficient conditions for the existence and hence equality of the limits are known, but the common limiting value itself is not explicitly characterized. In contrast, here we show that the one-sided limits of arbitrary moments of the double-layer traction field exist and are equal under a weaker set of sufficient conditions, and we additionally provide a novel expression for the common limiting value of the moments. For completeness, we also provide a result for the single-layer traction field.

Our results are applicable to general boundary integral formulations of Stokes flow problems in which the classic single- and double-layer potentials are employed. As an example, we outline a model for the Stokesian dynamics of an immersed flexible body with an elastic internal energy function and an arbitrary but finite number of internal degrees of freedom. Such a body may not only translate and rotate but also bend, twist, stretch, and otherwise deform as allowed by its parameterization. The hydrodynamic properties of such a body are encapsulated in a Stokes resistance operator defined by moments of the traction field with respect to appropriate weighting functions, and our results provide a convenient characterization of these moments. The Stokes resistance operator for filaments, membranes, and more general bodies, both rigid and flexible, is of interest in a variety of problems, ranging from the study of swimming motions of microorganisms to the optimal design of microrobots with potential applications such as targeted drug delivery; see, for example, [6, 12, 16, 21, 23, 24, 30, 32, 34] and the references therein.

The presentation is structured as follows. In section 2 we outline the Stokes equations for a fluid in a given domain. In section 3 we establish notation and outline the properties of the classic surface potentials for the Stokes equations that will be needed throughout our developments. In section 4 we state our main result, and in section 5 we describe an application of our result to the modeling of immersed flexible bodies. In section 6 we provide a proof of our result.

2. The Stokes equations. Here we outline the Stokes equations and introduce the flow quantities that will be the focus of our analysis. For further details on these equations, related boundary value problems, and associated potential theory, see [11, 20, 26, 29].

2.1. Domain, velocity, and pressure. We consider the steady motion of an incompressible viscous fluid at low Reynolds number in a given three-dimensional domain. We denote the domain of interest by $D_+ \subset \mathbb{R}^3$, the complementary domain by $D_- \subset \mathbb{R}^3$, and the boundary between them by $\mathit{\Gamma} \subset \mathbb{R}^3$. The Stokes equations for the fluid velocity field $u : D_+ \to \mathbb{R}^3$ and pressure field $p : D_+ \to \mathbb{R}$ are, in nondimensional form,

$$\begin{aligned} \Delta u(x) &= \nabla p(x), \\ \nabla \cdot u(x) &= 0, \end{aligned} \qquad \text{or} \qquad \begin{aligned} u_{i,jj}(x) &= p_{,i}(x), \\ u_{i,i}(x) &= 0, \end{aligned} \qquad x \in D_+. \tag{2.1}$$

We assume that $D_- \cup \mathit{\Gamma} \cup D_+$ fills all of three-dimensional space, that D_- and D_+ are open, and that D_+ is connected. Moreover, we assume that $\mathit{\Gamma}$ consists of a finite number of disjoint, closed, bounded, and orientable components, each of which is a Lyapunov surface [8]. Under these conditions, various boundary value problems can be formulated and studied using potential theoretic techniques. While D_+ could be either an interior or an exterior domain, we assume here and throughout that D_+ is the exterior and D_- is the interior domain with respect to $\mathit{\Gamma}$.

Unless mentioned otherwise, all vector quantities are referred to a single basis and indices take values from one to three. Moreover, we use the usual conventions that a pair of repeated indices implies summation and that indices appearing after a comma denote partial derivatives. We assume here and throughout that all quantities have been nondimensionalized using a characteristic length scale $\ell > 0$, velocity scale $\vartheta > 0$, and force scale $\mu\vartheta\ell > 0$, where μ is the absolute viscosity of the fluid. The dimensional quantities corresponding to $\{x, u, p\}$ are $\{\ell x, \vartheta u, \mu\vartheta\ell^{-1}p\}$.

2.2. Stress, traction, and moments. The stress field associated with a velocity-pressure pair (u, p) is a function $\sigma : D_+ \to \mathbb{R}^{3\times 3}$ defined by

$$\sigma_{ij}(x) = -p(x)\delta_{ij} + u_{i,j}(x) + u_{j,i}(x), \tag{2.2}$$

where δ_{ij} is the standard Kronecker delta symbol. For each $x \in D_+$ the stress tensor σ is symmetric in the sense that $\sigma_{ij} = \sigma_{ji}$. The traction field $f : S \to \mathbb{R}^3$ exerted by the fluid on a given oriented surface $S \subset D_+$ is defined by

$$f_i(x) = \sigma_{ij}(x)\nu_j(x), \tag{2.3}$$

where $\nu : S \to \mathbb{R}^3$ is a given unit normal field. The traction is the force per unit area exerted on S by the fluid on the positive side of S as determined by the normal. The resultant force $F \in \mathbb{R}^3$ and torque $T \in \mathbb{R}^3$, about an arbitrary point $\gamma \in \mathbb{R}^3$, associated with the traction field are

$$F_i = \int_S f_i(x)\, dA_x, \qquad T_i = \int_S \varepsilon_{ijk}(x_j - \gamma_j) f_k(x)\, dA_x, \tag{2.4}$$

where ε_{ijk} is the standard permutation symbol and dA_x denotes an infinitesimal area element at $x \in S$.

More generally, given an oriented surface S and a vector field $\eta : S \to \mathbb{R}^3$, we may define a stress or traction moment $\mathcal{L} \in \mathbb{R}$ by

$$\mathcal{L} = \int_S \eta_i(x) f_i(x)\, dA_x. \tag{2.5}$$

By inspection, we see that specific choices for η lead to the force and torque expressions outlined above. The characterization of the traction moment $\mathcal{L}$ for a given velocity-pressure pair (u, p), surface S, and moment function η is our main problem of interest; we will be specifically interested in the case when S is closed and bounded and tends to $\varGamma$ in an appropriate sense.

3. The Stokes potentials. Here we outline the classic single- and double-layer potentials which play a central role in the potential theory for the Stokes equations. We use ν to denote the outward unit normal field on $\varGamma$, and as before, D_- and D_+ denote the interior and exterior domains.

3.1. Velocity, pressure potentials. Let $\psi : \varGamma \to \mathbb{R}^3$ be a continuous function. Then by the Stokes single-layer velocity and pressure potentials on $\varGamma$ with density ψ we mean

$$\begin{aligned} V_i[\varGamma, \psi](x) &= \int_\varGamma E_V^{ij}(x, y) \psi_j(y)\, dA_y, \\ P_V[\varGamma, \psi](x) &= \int_\varGamma \varPi_V^{j}(x, y) \psi_j(y)\, dA_y, \end{aligned} \tag{3.1}$$

and by the Stokes double-layer velocity and pressure potentials on $\varGamma$ with density ψ we mean

$$\begin{aligned} W_i[\varGamma, \psi](x) &= \int_\varGamma E_W^{ijl}(x, y) \psi_j(y) \nu_l(y)\, dA_y, \\ P_W[\varGamma, \psi](x) &= \int_\varGamma \varPi_W^{jl}(x, y) \psi_j(y) \nu_l(y)\, dA_y. \end{aligned} \tag{3.2}$$

Here $(E_V^{ij}, \varPi_V^j)$ and $(E_W^{ijl}, \varPi_W^{jl})$ are fundamental-type solutions of the Stokes equations called the stokeslet and stresslet, respectively; they are solutions of the free-space Stokes equations with different types of singular forcing at the point y [7, 31]. Using the notation $z = x - y$ and $r = |z|$, explicit expressions for these solutions are

$$E_V^{ij}(x, y) = \frac{\delta_{ij}}{r} + \frac{z_i z_j}{r^3}, \quad \varPi_V^j(x, y) = \frac{2 z_j}{r^3}, \tag{3.3}$$

$$E_W^{ijl}(x, y) = \frac{3 z_i z_j z_l}{r^5}, \quad \varPi_W^{jl}(x, y) = -\frac{2\delta_{jl}}{r^3} + \frac{6 z_j z_l}{r^5}. \tag{3.4}$$

We remark that, due to the linearity of the free-space equations, the above solutions are defined up to an arbitrary choice of normalization. The choice of normalization naturally affects various constants in the developments that follow but is not crucial in any way; the choice adopted here is taken from [7]. While we only consider the Stokes potentials with densities in the classic spaces of continuous functions, they could also be considered on various Sobolev spaces [11].

3.2. Properties of velocity, pressure potentials. For any continuous density ψ, the potentials $(V[\Gamma,\psi], P_V[\Gamma,\psi])$ and $(W[\Gamma,\psi], P_W[\Gamma,\psi])$ are smooth at each $x \notin \Gamma$. Moreover, by virtue of their definitions as linear combinations of stokeslets and stresslets, they satisfy the Stokes equations (2.1) at each $x \notin \Gamma$. Because of this property, a given boundary value problem for the Stokes equations can be reduced to finding a density ψ that will produce given data on Γ; hence an understanding of the behavior of these and related potentials for points on and near Γ is essential.

The velocity potentials $V[\Gamma,\psi]$ and $W[\Gamma,\psi]$ are finite for all $x \in D_- \cup \Gamma \cup D_+$. In the special case when $x \in \Gamma$, both corresponding integrals are only weakly singular and hence exist as improper integrals in the usual sense [8] provided that Γ is a Lyapunov surface. The restrictions of $V[\psi,\Gamma]$ and $W[\psi,\Gamma]$ to Γ are denoted by $\overline{V}[\psi,\Gamma]$ and $\overline{W}[\psi,\Gamma]$. These restrictions are continuous functions on Γ; moreover, for any $x_0 \in \Gamma$, the following pointwise limit relations hold [20]:

$$\lim_{\substack{x\to x_0\\ x\in D_+}} V[\Gamma,\psi](x) = \overline{V}[\Gamma,\psi](x_0), \tag{3.5}$$

$$\lim_{\substack{x\to x_0\\ x\in D_-}} V[\Gamma,\psi](x) = \overline{V}[\Gamma,\psi](x_0), \tag{3.6}$$

$$\lim_{\substack{x\to x_0\\ x\in D_+}} W[\Gamma,\psi](x) = 2\pi\psi(x_0) + \overline{W}[\Gamma,\psi](x_0), \tag{3.7}$$

$$\lim_{\substack{x\to x_0\\ x\in D_-}} W[\Gamma,\psi](x) = -2\pi\psi(x_0) + \overline{W}[\Gamma,\psi](x_0). \tag{3.8}$$

Notice that, by continuity of ψ and $\overline{W}[\Gamma,\psi]$, the one-sided limits in (3.7) and (3.8) are themselves continuous functions on Γ. Standard arguments [8] can be used to show that all four of the above limits are approached uniformly in $x_0 \in \Gamma$.

In contrast, the pressure potentials $P_V[\Gamma,\psi]$ and $P_W[\Gamma,\psi]$ are not as simple. These potentials are finite for all $x \in D_- \cup D_+$, but when $x \in \Gamma$, the corresponding integrals are singular and hyper-singular, respectively, and hence do not exist as improper integrals in the usual sense. Nevertheless, when the density ψ and surface Γ are sufficiently regular, the potentials $P_V[\Gamma,\psi]$ and $P_W[\Gamma,\psi]$ have pointwise limits as x approaches Γ [20, 29, 35]. Notice that the existence of such limits is intimately connected with regularity up to the boundary of solutions of the Stokes equations.

3.3. Stress, traction potentials. The single- and double-layer stress potentials associated with $(V[\Gamma,\psi], P_V[\Gamma,\psi])$ and $(W[\Gamma,\psi], P_W[\Gamma,\psi])$, respectively, are

$$\Sigma_V^{ik}[\Gamma,\psi](x) = \int_\Gamma \Xi_V^{ikj}(x,y)\psi_j(y)\, dA_y, \tag{3.9}$$

$$\Sigma_W^{ik}[\Gamma,\psi](x) = \int_\Gamma \Xi_W^{ikjl}(x,y)\psi_j(y)\nu_l(y)\, dA_y, \tag{3.10}$$

where Ξ_V^{ikj} and Ξ_W^{ikjl} are the stress fields corresponding to the stokeslet and stresslet solutions in (3.3) and (3.4). In particular, we have [7, 31]

$$\Xi_V^{ikj}(x,y) = -\frac{6z_iz_kz_j}{r^5}, \tag{3.11}$$

$$\Xi_W^{ikjl}(x,y) = \frac{2\delta_{ik}\delta_{jl}}{r^3} + \frac{3(\delta_{ij}z_kz_l + \delta_{il}z_jz_k + \delta_{jk}z_iz_l + \delta_{lk}z_iz_j)}{r^5} - \frac{30z_iz_jz_kz_l}{r^7}. \tag{3.12}$$

The limiting traction fields on Γ associated with $\Sigma_V[\Gamma,\psi]$ and $\Sigma_W[\Gamma,\psi]$ are of particular interest. To study these, we consider a neighborhood of Γ, with points parameterized as $x_\tau = x_0 + \tau\nu(x_0)$, where $x_0 \in \Gamma$ and $\tau \in [-c,c]$ for some $c > 0$ sufficiently small, and we extend the normal field such that $\nu(x_\tau) = \nu(x_0)$. In this neighborhood, we have the traction potentials

$$f_V^i[\Gamma,\psi](x_\tau) = \Sigma_V^{ik}[\Gamma,\psi](x_\tau)\nu_k(x_\tau), \quad f_W^i[\Gamma,\psi](x_\tau) = \Sigma_W^{ik}[\Gamma,\psi](x_\tau)\nu_k(x_\tau). \tag{3.13}$$

3.4. Properties of stress, traction potentials. For any continuous density ψ, the stress potentials $\Sigma_V[\Gamma,\psi]$ and $\Sigma_W[\Gamma,\psi]$ are smooth at each $x \notin \Gamma$. At any such point, these potentials provide an explicit expression for the stresses associated with the velocity-pressure pairs $(V[\Gamma,\psi], P_V[\Gamma,\psi])$ and $(W[\Gamma,\psi], P_W[\Gamma,\psi])$. Moreover, for any fixed $\tau \neq 0$, the set of points x_τ define a surface Γ_τ that is parallel to Γ and has the same normal field, and the traction potentials $f_V[\Gamma,\psi]$ and $f_W[\Gamma,\psi]$ provide explicit expressions for the associated tractions on this surface.

The single-layer traction potential $f_V[\Gamma,\psi]$ is finite for all $x_0 \in \Gamma$ and $\tau \in [-c,c]$. Indeed, in view of (3.11) and (3.4), we see that the single-layer traction potential is closely related to the double-layer velocity potential and consequently has similar properties. Specifically, when $\tau = 0$, the corresponding integral is only weakly singular and hence exists as an improper integral provided that Γ is a Lyapunov surface. The restriction of $f_V[\Gamma,\psi]$ to Γ ($\tau = 0$) is denoted by $\overline{f}_V[\Gamma,\psi]$. This restriction is a continuous function on Γ; moreover, for any $x_0 \in \Gamma$, the following pointwise limit relations hold [20, 29]:

$$\lim_{\tau\to 0^+} f_V[\Gamma,\psi](x_\tau) = -4\pi\psi(x_0) + \overline{f}_V[\Gamma,\psi](x_0), \tag{3.14}$$

$$\lim_{\tau\to 0^-} f_V[\Gamma,\psi](x_\tau) = \quad 4\pi\psi(x_0) + \overline{f}_V[\Gamma,\psi](x_0). \tag{3.15}$$

Similar to before, standard arguments [8] can be used to show that both of the above limits are approached uniformly in $x_0 \in \Gamma$.

In contrast, the double-layer traction potential $f_W[\Gamma,\psi]$ is not as simple. This potential is finite for all $x_0 \in \Gamma$ and $\tau \in [-c,0) \cup (0,c]$, but when $\tau = 0$, the corresponding integral is hyper-singular and hence does not exist as an improper integral in the usual sense. Nevertheless, when the density ψ and surface Γ are sufficiently regular, the potential $f_W[\Gamma,\psi]$ has pointwise limits as $\tau \to 0^\pm$, and moreover, for each $x_0 \in \Gamma$, the following equality holds [29]:

$$\lim_{\tau\to 0^+} f_W[\Gamma,\psi](x_\tau) = \lim_{\tau\to 0^-} f_W[\Gamma,\psi](x_\tau). \tag{3.16}$$

The above result is commonly referred to as the Lyapunov–Tauber theorem. Although elegant, this result does not provide an explicit or easily computable expression for the limiting double-layer traction field. Sufficient conditions for the existence and hence equality of the above pointwise limits are that $\psi \in C^{1,\gamma}(\Gamma,\mathbb{R}^3)$ for some $0 < \gamma \leq 1$, which means that the density is a differentiable function from Γ into $\mathbb{R}^3$ with Hölder continuous derivatives, and that $\Gamma \in C^{2,\gamma}$, which means that the surface, when considered locally as a graph in Cartesian coordinates, is twice differentiable with Hölder continuous second derivatives [29].

4. Main result. Consider a neighborhood of Γ, with points parameterized as $x_\tau = x_0 + \tau\nu(x_0)$, where $x_0 \in \Gamma$ and $\tau \in [-c,c]$ as before. Let $f_V[\Gamma,\psi]$ and $f_W[\Gamma,\psi]$ be the single- and double-layer traction potentials, and given any vector field η on Γ, consider an extension into the neighborhood defined by $\eta(x_\tau) = \eta(x_0)$. Moreover, for any fixed τ, let Γ_τ be the surface parallel to Γ defined by the points x_τ, and for any function $g(x)$, let $\widehat{g}(x,y) = g(x) - g(y)$. In view of (2.5), consider the traction moments

$$\mathcal{L}_V^\tau = \int_{\Gamma_\tau} \eta_i(x_\tau) f_V^i[\Gamma,\psi](x_\tau)\, dA_{x_\tau}, \tag{4.1}$$

$$\mathcal{L}_W^\tau = \int_{\Gamma_\tau} \eta_i(x_\tau) f_W^i[\Gamma,\psi](x_\tau)\, dA_{x_\tau}. \tag{4.2}$$

THEOREM 4.1. *Let Γ be a closed, bounded Lyapunov surface with outward unit normal field ν. If $\Gamma \in C^{1,1}$, $\psi \in C^{0,1}(\Gamma,\mathbb{R}^3)$, and $\eta \in C^{0,\gamma}(\Gamma,\mathbb{R}^3)$ for $0 < \gamma \le 1$, then the limiting values of $\mathcal{L}_V^\tau$ and $\mathcal{L}_W^\tau$ exist and*

$$\lim_{\tau\to 0^+} \mathcal{L}_V^\tau = -4\pi \int_\Gamma \eta_i(x)\psi_i(x)\, dA_x + \int_\Gamma \int_\Gamma K_V[\eta,\psi](x,y)\, dA_x dA_y, \tag{4.3}$$

$$\lim_{\tau\to 0^-} \mathcal{L}_V^\tau = \quad 4\pi \int_\Gamma \eta_i(x)\psi_i(x)\, dA_x + \int_\Gamma \int_\Gamma K_V[\eta,\psi](x,y)\, dA_x dA_y, \tag{4.4}$$

$$\lim_{\tau\to 0^+} \mathcal{L}_W^\tau = \int_\Gamma \int_\Gamma K_W[\eta,\psi](x,y)\, dA_x dA_y = \lim_{\tau\to 0^-} \mathcal{L}_W^\tau, \tag{4.5}$$

where

$$K_V[\eta,\psi](x,y) = \eta_i(x) \varXi_V^{ikj}(x,y)\psi_j(y)\nu_k(x) \tag{4.6}$$

and

$$\begin{aligned} K_W[\eta,\psi](x,y) = \frac{1}{2}\widehat{\eta}_i(x,y)\varXi_W^{ikjl}(x,y)\Big[&\widehat{\psi}_j(y,x)\nu_l(y)\nu_k(x) \\ &+ \psi_j(x)\widehat{\nu}_l(y,x)\nu_k(x) + \psi_j(x)\nu_l(x)\widehat{\nu}_k(x,y)\Big]. \end{aligned} \tag{4.7}$$

Thus, under appropriate assumptions, the limiting values of the single- and double-layer traction moments $\mathcal{L}_V^\tau$ and $\mathcal{L}_W^\tau$ exist and can be expressed as weakly singular integrals depending only pointwise on the density ψ and weight η. The fact that the integral for the double-layer case in (4.5) is only weakly singular is a consequence of the factors $\widehat{\eta}$, $\widehat{\psi}$, and, $\widehat{\nu}$ in (4.7). The results for the single-layer moments are essentially local; they do not rely on the closedness of Γ and are based on the explicit pointwise limit relations in (3.14) and (3.15). These results are straightforward and are included only for completeness and purposes of comparison. In contrast, the results for the double-layer moments are essentially nonlocal; they rely on the closedness of Γ and are not based on explicit pointwise limit relations. Notice that, since the results hold for arbitrary weighting functions, we may consider those with localized support and thereby obtain localized information on the double-layer traction field.

We point out that the results for the double-layer moments are obtained without any explicit application of Stokes theorem or integration by parts, as have been considered in other contexts [11, 25], which would generally lead to expressions involving surface derivatives of ψ or η. Moreover, while the double-layer results are consistent with the Lyapunov–Tauber theorem in (3.16) regarding the equality of pointwise limits, the results are not implied by this relation but instead are of a different nature and contain important additional information. Specifically, the above results state that the one-sided limits of the double-layer moments exist and are equal, and additionally they provide a straightforward, explicit expression for the common limiting value of the moments. Moreover, as compared to the Lyapunov–Tauber theorem, the above results hold under milder regularity conditions on the density and surface. These milder conditions seem rather sharp for the technique of proof and classic function spaces employed here; whether or not the results would hold under weaker conditions is an open question.

All the above results have application to general boundary integral formulations of Stokes flow problems in which the classic single- and double-layer potentials are employed. One application would be the determination of generalized forces associated with the coupled translational, rotational, and shape dynamics of immersed flexible bodies. In this case, a number of different boundary integral formulations could be used to represent the flow and, independent of the interpretation of the density, the above results would provide the generalized forces. The case of immersed rigid bodies is simpler and can be approached in various ways [7, 14, 15, 28].

5. Application to flexible bodies. Here we briefly illustrate how the results in Theorem 4.1 arise in the modeling of immersed flexible bodies. As before, we use Γ to denote a bounded surface with associated interior domain D_- and exterior domain D_+.

5.1. Kinematics. We consider a body with current configuration D_- and reference configuration D_0 and suppose points $x \in D_-$ are in bijective correspondence with points $\mathsf{x} \in \mathsf{D}_0$ via a parameterized map of the form

$$x = \Phi(\mathsf{x}; \gamma, G, q) = \gamma + \phi_i(\mathsf{x}, q) g_i, \tag{5.1}$$

where $\gamma \in \mathbb{R}^3$ is a vector that defines the position of a body origin, $G = (g_1, g_2, g_3) \in \mathbb{R}^{3\times 3}$ is an orthonormal matrix with column vectors $g_i \in \mathbb{R}^3$ $(i = 1, 2, 3)$ that define the orientation of a body frame, $q \in \mathbb{R}^m$ is a vector of shape parameters, and $\phi(\mathsf{x}, q) \in \mathbb{R}^3$ is a given function which defines the shape of the body in the body frame. For example, D_0 may be a straight cylinder, and D_- may be a curved cylinder, where the curvature and torsion of the axial curve, along with other features such as the radius and twist of the cylinder about its axial curve, are defined by the parameters in q. Alternatively, and more directly, the parameters in q could be the positions of a set of nodes which together with an interpolation rule define the axial curve of the cylinder. For simplicity we restrict our attention to bodies whose shapes have a finite number of degrees of freedom in this sense.

A motion of the body is a one-parameter family of configurations defined by $x = \Phi(\mathsf{x}; \gamma(t), G(t), q(t))$. For each fixed $t \geq 0$, the velocity of the body is described by a vector field $u : D_- \to \mathbb{R}^3$ of the form

$$u(x) = v + \omega \times (x - \gamma) + \eta_\alpha(x)\vartheta_\alpha, \tag{5.2}$$

where $v \in \mathbb{R}^3$ and $\omega \in \mathbb{R}^3$ are the linear and angular velocities of the body frame, $\vartheta = \dot{q} \in \mathbb{R}^m$ is the rate of change of the shape parameters, and $\eta_\alpha(x) = (\partial\phi_i/\partial q_\alpha)(\mathsf{x}, q) g_i \in \mathbb{R}^3$ $(\alpha = 1, \dots, m)$ are vector fields which form a natural basis for the shape dynamics. Here we exploit the fact that any function of $\mathsf{x} \in \mathsf{D}_0$ can be considered as a function of $x \in D_-$ and conversely; as before, we employ the summation convention on pairs of repeated indices, where the range of the sum is determined by the context. Thus the configuration of the body is parameterized by (γ, G, q) and its velocity field by (v, ω, ϑ).

5.2. Load balance. For each fixed $t \geq 0$, we suppose that the body is subject to a set of external forces $f^{(\beta)} \in \mathbb{R}^3$ at specified material points $x^{(\beta)} = \Phi(\mathsf{x}^{(\beta)}; \gamma, G, q)$

($\beta = 1, \dots, n$) and an external hydrodynamic traction field per unit area $f : \Gamma \to \mathbb{R}^3$; the velocity and natural basis vector at each $x^{(\beta)}$ are denoted by $u^{(\beta)}$ and $\eta_\alpha^{(\beta)}$. We also suppose that the body is elastic, with an internal energy function $E(q) \in \mathbb{R}$. For quasi-static motions in which the kinetic energy of the body can be neglected, the power expended by the external loads is balanced by the rate of change of internal energy, namely,

$$\sum_\beta f^{(\beta)} \cdot u^{(\beta)} + \int_\Gamma f \cdot u \, dA = \frac{d}{dt} E(q). \tag{5.3}$$

An external body force field per unit mass could also be included in the above balance; to do so would require the consideration of hydrostatic effects since the traction we consider is purely hydrodynamic, and also the mass density and volumetric rate of strain fields within the body, but we do not pursue that here.

Substitution of (5.2) into (5.3), together with the fact that (5.3) must hold for motions with arbitrary values of (v, ω, ϑ), leads to the load balance relations

$$\begin{aligned} F^{\text{ext}} + F^{\text{hyd}} = 0, \qquad T^{\text{ext}} + T^{\text{hyd}} = 0, \\ H_\alpha^{\text{ext}} + H_\alpha^{\text{hyd}} = H_\alpha^{\text{int}}, \quad \alpha = 1, \dots, m. \end{aligned} \tag{5.4}$$

The first two equations above are the usual balance of force and torque relations that arise in the case of rigid bodies, where the reference point for the torque is the body origin. The third equation is a balance of generalized forces relation that arises in the case of a flexible body as considered here. Specifically, we have

$$H_\alpha^{\text{int}} = \frac{\partial E}{\partial q_\alpha}, \tag{5.5}$$

$$F^{\text{ext}} = \sum_\beta f^{(\beta)}, \quad T^{\text{ext}} = \sum_\beta (x^{(\beta)} - \gamma) \times f^{(\beta)}, \quad H_\alpha^{\text{ext}} = \sum_\beta \eta_\alpha^{(\beta)} \cdot f^{(\beta)}, \tag{5.6}$$

$$F^{\text{hyd}} = \int_\Gamma f \, dA, \quad T^{\text{hyd}} = \int_\Gamma (x - \gamma) \times f \, dA, \quad H_\alpha^{\text{hyd}} = \int_\Gamma \eta_\alpha \cdot f \, dA. \tag{5.7}$$

5.3. Hydrodynamics. For each fixed $t \geq 0$, we suppose the body is immersed in an incompressible viscous fluid at rest at infinity, and we model the hydrodynamic traction field $f : \Gamma \to \mathbb{R}^3$ using the Stokes system (2.1) in the exterior domain D_+, with no-slip data of the form (5.2) on the boundary Γ, together with conditions of vanishing velocity and pressure at infinity. Linearity of (2.1)–(2.3), together with (5.2) and (5.7), implies the existence of a linear map from (v, ω, ϑ) to $(F^{\text{hyd}}, T^{\text{hyd}}, H^{\text{hyd}})$, where $H^{\text{hyd}} = (H_\alpha^{\text{hyd}}) \in \mathbb{R}^m$ is the generalized force vector. We denote this map or operator by $-R \in \mathbb{R}^{(6+m)\times(6+m)}$ so that

$$(F^{\text{hyd}}, T^{\text{hyd}}, H^{\text{hyd}}) = -R(v, \omega, \vartheta). \tag{5.8}$$

The evaluation of the entries of R requires the evaluation of the hydrodynamic traction moments $(F^{\text{hyd}}, T^{\text{hyd}}, H^{\text{hyd}})$ in (5.7) for independent choices of the velocities (v, ω, ϑ) in (5.2). The results contained in Theorem 4.1 provide a basic tool for this purpose; the results are especially useful for the evaluation of the components H_α^{hyd} of the generalized force. For example, for given (v, ω, ϑ), the Stokes boundary value problem could be solved using any combination of single- and double-layer potentials on Γ with density ψ. Once ψ is obtained, the traction moments for different weighting functions η_α are then obtained from the theorem: the contribution from the single-layer potential is given by (4.3), which is the limiting value of (4.1) (the limit from the exterior domain is appropriate), and the contribution from the double-layer potential is given by (4.5), which is the limiting value of (4.2).

5.4. Equations of motion. Combining the kinematic, load balance, and hydrodynamic relations leads to the following equations for the quasi-static motion of an immersed flexible body:

$$\begin{gathered}\dot{\gamma} = v, \qquad \dot{g}_i = \omega \times g_i, \qquad \dot{q} = \vartheta, \\ R(v, \omega, \vartheta) = (F^{\text{ext}}, T^{\text{ext}}, H^{\text{ext}} - H^{\text{int}}).\end{gathered} \tag{5.9}$$

Here H^{int} and $(F^{\text{ext}}, T^{\text{ext}}, H^{\text{ext}})$ are loads as defined in (5.5) and (5.6), and R is the Stokes resistance operator defined by the Stokes equations in the domain exterior to the body; the loads and resistance operator are generally configuration dependent. Notice that velocities can be determined from specified loads, and more generally, various mixed subsets of velocity and load components can be determined from specified values of the remaining components. The above equations can be used to model the coupled motions of a body that may translate and rotate, and also bend, twist, stretch, and otherwise deform as allowed by its parameterization. Such motions may be of interest in the study of the sedimentation and transport of particles and the propulsion of microorganisms and micro-robots in a number of applications where the assumption of rigidity may not be appropriate or desirable; see, for example, [6, 12, 16, 21, 23, 24, 30, 32, 34].

6. Proof. In this section we provide a proof of Theorem 4.1. We use the same notation and conventions as in previous sections. Moreover, we use C to denote a generic positive constant whose value may change from one appearance to the next, and we use $|\cdot|$ to denote a Euclidean norm or measure of a surface, as determined by the context. Furthermore, we omit indices on vector and tensor quantities whenever there is no cause for confusion. With the exception of intervals on the real line, the delimiters $[\cdot]$ and $(\cdot)$ are used interchangeably; there is no implication of a jump operator or otherwise.

6.1. Preliminaries. By assumption, $\Gamma \subset \mathbb{R}^3$ consists of a finite number of disjoint, closed, bounded, and orientable components, each of which is a Lyapunov surface [8]. Thus,

(L1) there exists an outward unit normal $\nu(x)$ and tangent plane $T_x\Gamma$ at every $x \in \Gamma$,

(L2) there exists constants $C > 0$ and $0 < \lambda \leq 1$ such that $\theta(\nu(x), \nu(y)) \leq C|x-y|^\lambda$ for all $x, y \in \Gamma$, where $\theta(\nu(x), \nu(y))$ is the angle between $\nu(x)$ and $\nu(y)$,

(L3) there exists a constant $d > 0$ such that, for every $x \in \Gamma$, the subset $\Gamma \cap B(x, d)$ is a graph over $T_x\Gamma$, where $B(x, d)$ is the closed ball of radius d with center at x.

We refer to λ and d as a Lyapunov exponent and radius associated with Γ, and for any $x \in \Gamma$ we refer to $B(x, d)$ as a Lyapunov ball at x. Notice that if (L2) and (L3) hold for some values of λ and d, then they also hold for all smaller values. We assume that values for λ and d are fixed once and for all and assume specifically that $\lambda = 1$.

For any $x \in \Gamma$, we use $\Gamma_{x,d}$ to denote the portion of Γ within the Lyapunov ball at x, and we use $\Omega_{x,d}$ to denote the image of $\Gamma_{x,d}$ on $T_x\Gamma$ under projection parallel to $\nu(x)$. We refer to $\Gamma_{x,d}$ as the Lyapunov patch at x. Without loss of generality, we identify $\Omega_{x,d}$ with a subset of $\mathbb{R}^2$ and identify x with the origin. We reserve the notation $T_x\Gamma$ to indicate the tangent plane considered as a subspace of $\mathbb{R}^3$. The Lyapunov condition (L3) implies that the map

$$y = \varphi_x(\zeta), \quad y \in \Gamma_{x,d}, \quad \zeta \in \Omega_{x,d}, \tag{6.1}$$

defined by projection parallel to $\nu(x)$, is a bijection. We refer to $y = \varphi_x(\zeta)$ with inverse $\zeta = \varphi_x^{-1}(y)$ as a local Cartesian parameterization of Γ at x. Notice that, for any $x \in \Gamma$, the local Cartesian coordinates are uniquely defined up to the choice of orthonormal basis in $T_x\Gamma$.

We say that Γ is of class $C^{1,1}$, and use the notation $\Gamma \in C^{1,1}$, if the local Cartesian map φ_x and its inverse φ_x^{-1} are of class $C^{1,1}$ in their respective domains for each x and additionally have Lipschitz constants that are uniform in x. We will also have need to consider a local polar parameterization of Γ at x of the form

$$y = \varphi_x^{\text{polar}}(\rho, \theta), \quad y \in \Gamma_{x,d}, \quad (\rho, \theta) \in \Omega_{x,d}^{\text{polar}}, \tag{6.2}$$

where $\Omega_{x,d}^{\text{polar}}$ is a subset of $\mathbb{R}_+ \times [0, 2\pi)$. For a given choice of orthonormal basis in $T_x\Gamma$, the polar parameterization is defined from the Cartesian one using the relation $\zeta = (\rho\cos\theta, \rho\sin\theta)$. The inverse map $(\rho, \theta) = \varphi_x^{\text{polar},-1}(y)$ is defined for all $y \in \Gamma_{x,d}$, $y \neq x$. The relation between an area element dA_y at $y \in \Gamma_{x,d}$ and the corresponding element dA_ζ at $\zeta = \varphi_x^{-1}(y) \in \Omega_{x,d}$ is given by

$$dA_y = j_x(\zeta)\, dA_\zeta = j_x(\rho\cos\theta, \rho\sin\theta)\, \rho\, d\rho\, d\theta, \tag{6.3}$$

where j_x is the Jacobian associated with the coordinate map φ_x. When the Lyapunov radius d is chosen sufficiently small, the condition in (L2) implies the following uniform bounds for all $x \in \Gamma$ and $y = \varphi_x(\zeta) \in \Gamma_{x,d}$ and some fixed constant C:

$$1 \leq j_x(\zeta) \leq 1 + C\rho \leq 2, \quad \rho \leq |x - y| \leq \rho + C\rho^3 \leq 2\rho. \tag{6.4}$$

Throughout our analysis, we consider a neighborhood of Γ, with points parameterized by the map

$$\xi = \vartheta_\tau(x) := x + \tau\nu(x), \quad x \in \Gamma, \quad \tau \in [-c, c]. \tag{6.5}$$

Provided the constant $c > 0$ is sufficiently small, the image $\Gamma_\tau = \vartheta_\tau(\Gamma)$ is a surface parallel to Γ for each fixed τ and is also Lyapunov and of class $C^{1,1}$. From the geometry of parallel surfaces, for any $\tau \in [-c, c]$, $x \in \Gamma$, and $\xi = \vartheta_\tau(x) \in \Gamma_\tau$ we have the following classic result (see, for example, [27]):

$$\nu(\xi) = \nu(x), \quad dA_\xi = J_\tau(x)\, dA_x, \quad J_\tau(x) = 1 + 2\tau\kappa^m(x) + \tau^2\kappa^g(x). \tag{6.6}$$

Here $\nu(\xi)$ and $\nu(x)$ are the outward unit normals on Γ_τ and Γ, dA_ξ and dA_x are area elements on Γ_τ and Γ, and κ^m and κ^g are the mean and Gaussian curvatures of Γ. Here we use the convention that curvature is positive when Γ curves away from its outward unit normal. We denote the inverse of $\xi = \vartheta_\tau(x)$ by $x = \vartheta_\tau^{-1}(\xi)$. In view of (6.5) and $(6.6)_1$ we have $x = \xi - \tau\nu(\xi)$. The assumption that $\Gamma \in C^{1,1}$ implies that the curvatures κ^m and κ^g are defined almost everywhere and are uniformly bounded on Γ. Moreover, when the Lyapunov radius d is chosen sufficiently small, the condition in (L2) implies the following uniform bounds for all $x \in \Gamma$, $y \in \Gamma_{x,d}$, and $\tau \in [-c, c]$, where $r = |x - y|$:

$$\tfrac{r^2+\tau^2}{2} \leq |y - \vartheta_\tau(x)|^2 \leq \tfrac{3(r^2+\tau^2)}{2}, \quad \tfrac{r^2+\tau^2}{2} \leq |\vartheta_\tau(y) - x|^2 \leq \tfrac{3(r^2+\tau^2)}{2}. \tag{6.7}$$

6.2. Lemmata. We begin with a collection of estimates regarding certain types of integrals that will arise in our analysis. The results in (6.8) are simple generalizations of standard estimates for weakly singular integrals. The results in (6.9) are more delicate and their validity depends crucially on the translation parameter τ. The results follow from straightforward calculus; the details are omitted for brevity.

LEMMA 6.1. *Let $\Gamma \in C^{1,1}$ and $\alpha > 0$ be given, and let $d > 0$ be the Lyapunov radius of Γ. Then there exists constants $d_* \in (0, d]$ and $C > 0$ such that for all $x \in \Gamma$, $\tau \in [-c, c]\backslash\{0\}$, and $a \in (0, d_*]$ we have*

$$\int_{\Gamma_{x,a}} \frac{1}{(r^2+\tau^2)^{(2-\alpha)/2}}\, dA_y \le Ca^\alpha, \qquad \int_{\Gamma_{x,a}} \frac{r^{1+\alpha}}{(r^2+\tau^2)^{3/2}}\, dA_y \le Ca^\alpha, \tag{6.8}$$

$$\int_{\Gamma_{x,a}} \frac{r^\alpha|\tau|}{(r^2+\tau^2)^{3/2}}\, dA_y \le Ca^\alpha, \qquad \int_{\Gamma_{x,a}} \frac{|\tau|}{(r^2+\tau^2)^{3/2}}\, dA_y \le C. \tag{6.9}$$

Our next result establishes an important identity for the double-layer stress potential defined in (3.10). Let $\tau \in [-c, c]$ and functions $\psi, \eta : \Gamma \to \mathbb{R}^3$ be given, and consider corresponding functions $\psi, \eta : \Gamma_\tau \to \mathbb{R}^3$ defined by $\psi(\xi) = \psi(x)$ and $\eta(\xi) = \eta(x)$, where $\xi = \vartheta_\tau(x)$. Moreover, let $\Sigma_W[\Gamma, \psi]$ be the double-layer stress potential with density ψ on Γ, and let $\Sigma_W[\Gamma_\tau, \psi]$ be the corresponding potential with density ψ on Γ_τ. For $\tau \neq 0$, notice that $\Sigma_W[\Gamma, \psi]$ is finite at all points of Γ_τ and $\Sigma_W[\Gamma_\tau, \psi]$ is finite at all points of Γ, since $\Gamma_\tau \cap \Gamma = \emptyset$, and consider the moments defined by

$$M^\tau[\eta, \psi] = \int_{\Gamma_\tau} \eta_i(\xi)\Sigma_W^{ik}[\Gamma, \psi](\xi)\nu_k(\xi)\, dA_\xi, \tag{6.10}$$

$$N^\tau[\eta, \psi] = \int_{\Gamma} \eta_i(y)\Sigma_W^{ik}[\Gamma_\tau, \psi](y)\nu_k(y)\, dA_y. \tag{6.11}$$

Moreover, let $K_W[\eta, \psi]$ be the kernel defined in (4.7), and for $\tau \neq 0$ notice that $K_W[\eta, \psi](\xi, y)$ is finite for all $\xi \in \Gamma_\tau$ and $y \in \Gamma$.

LEMMA 6.2. *Let $\Gamma \in C^{1,1}$, $\psi \in C^0(\Gamma, \mathbb{R}^3)$, and $\eta \in C^0(\Gamma, \mathbb{R}^3)$ be given. Then for every $\tau \in [-c, c]\backslash\{0\}$ we have*

$$M^\tau[\eta, \psi] + N^\tau[\eta, \psi] = 2\int_{\Gamma_\tau}\int_{\Gamma} K_W[\eta, \psi](\xi, y)\, dA_y dA_\xi. \tag{6.12}$$

Proof. Let $A(x)$, $B(x)$, $C(x)$, and $D(x)$ be arbitrary functions and consider the convenient notation $A_\xi = A(\xi)$, $B_y = B(y)$, and so on. Then the following identity holds:

$$\begin{aligned} &A_\xi B_y C_y D_\xi + A_y B_\xi C_\xi D_y \\ &= (A_\xi - A_y)\Big[(B_y - B_\xi)C_y D_\xi + B_\xi(C_y - C_\xi)D_\xi + B_\xi C_\xi(D_\xi - D_y)\Big] \\ &\quad + A_y B_y C_y D_\xi + A_\xi B_\xi C_\xi D_y. \end{aligned} \tag{6.13}$$

Considering the above identity with $A = \eta_i$, $B = \psi_j$, $C = \nu_l$, and $D = \nu_k$, and contracting it with Ξ_W^{ikjl}, and using the definition of $K_W[\eta, \psi]$ in (4.7), we get

$$\begin{aligned} &\eta_i(\xi)\Xi_W^{ikjl}(\xi, y)\psi_j(y)\nu_l(y)\nu_k(\xi) + \eta_i(y)\Xi_W^{ikjl}(\xi, y)\psi_j(\xi)\nu_l(\xi)\nu_k(y) \\ &\quad = 2K_W[\eta, \psi](\xi, y) + \eta_i(y)\Xi_W^{ikjl}(\xi, y)\psi_j(y)\nu_l(y)\nu_k(\xi) \\ &\qquad + \eta_i(\xi)\Xi_W^{ikjl}(\xi, y)\psi_j(\xi)\nu_l(\xi)\nu_k(y). \end{aligned} \tag{6.14}$$

The above expression is finite for any $\tau \neq 0$ and can be integrated over $\xi \in \Gamma_\tau$ and $y \in \Gamma$. Performing the integration, and using (3.10), together with the symmetry property $\Xi_W^{ikjl}(\xi, y) = \Xi_W^{ikjl}(y, \xi)$ implied by (3.12), and using also (6.10) and (6.11), we find

$$\begin{aligned} M^\tau[\eta,\psi] + N^\tau[\eta,\psi] = 2\int_{\Gamma_\tau}\int_\Gamma & K_W[\eta,\psi](\xi,y)\, dA_y dA_\xi \\ & + \int_\Gamma \eta_i(y)\psi_j(y)\nu_l(y)\left[\int_{\Gamma_\tau} \Xi_W^{ikjl}(y,\xi)\nu_k(\xi)\, dA_\xi\right] dA_y \\ & + \int_{\Gamma_\tau} \eta_i(\xi)\psi_j(\xi)\nu_l(\xi)\left[\int_\Gamma \Xi_W^{ikjl}(\xi,y)\nu_k(y)\, dA_y\right] dA_\xi. \end{aligned} \tag{6.15}$$

The desired result follows from the fact that each of the two integrals in brackets on the right-hand side of the above expression vanishes. Indeed, the first integral in brackets is the resultant force on the closed surface Γ_τ generated by the fundamental stresslet solution (E_W^{ijl}, Π_W^{jl}) with pole at $y \notin \Gamma_\tau$, which vanishes by properties of the stresslet solution; see, for example, [7]. Similarly, the second integral in brackets is the resultant force on the closed surface Γ generated by the fundamental stresslet solution with pole at $\xi \notin \Gamma$, which vanishes by the same properties. ∎

The next result shows that the one-sided limits of the sum $M^\tau[\eta,\psi] + N^\tau[\eta,\psi]$, as $\tau \to 0^\pm$, exist and are equal under additional Hölder conditions on the functions ψ and η. Moreover, the common value of the one-sided limits is given by the two-fold integral of the kernel $K_W[\eta,\psi]$, which is weakly singular on Γ under the additional Hölder conditions.

LEMMA 6.3. *Let $\Gamma \in C^{1,1}$, $\psi \in C^{0,\alpha}(\Gamma, \mathbb{R}^3)$, and $\eta \in C^{0,\gamma}(\Gamma, \mathbb{R}^3)$ be given, where $\alpha > 0$ and $\gamma > 0$ such that $\alpha + \gamma > 1$. Then*

$$\lim_{\tau \to 0^\pm}\Big[M^\tau[\eta,\psi] + N^\tau[\eta,\psi]\Big] = 2\int_\Gamma\int_\Gamma K_W[\eta,\psi](x,y)\, dA_y dA_x. \tag{6.16}$$

Proof. For any $y \in \Gamma$ and $\tau \in [-c, c]\backslash\{0\}$ let

$$\overline{U}(y) = \int_\Gamma K_W[\eta,\psi](x,y)\, dA_x, \tag{6.17}$$

$$U(y,\tau) = \int_{\Gamma_\tau} K_W[\eta,\psi](\xi,y)\, dA_\xi = \int_\Gamma K_W[\eta,\psi](\vartheta_\tau(x),y)J_\tau(x)\, dA_x. \tag{6.18}$$

We first seek to show that $U(y,\tau) \to \overline{U}(y)$ as $\tau \to 0^\pm$ uniformly in y everywhere on Γ. To this end, notice that $U(y,\tau)$ is finite for all $y \in \Gamma$ and $\tau \in [-c,c]\backslash\{0\}$ since $\Gamma_\tau \cap \Gamma = \emptyset$. Moreover, from the conditions that $\nu \in C^{0,1}$, $\psi \in C^{0,\alpha}$, and $\eta \in C^{0,\gamma}$ on Γ, and the fact that these functions are constant in the normal direction, and the inequality $|y - x| \leq \sqrt{2}|y - \vartheta_\tau(x)|$ which follows from (6.7), we find that the quantities

$$\frac{|\nu(y) - \nu(\xi)|}{|y-\xi|}, \quad \frac{|\psi(y) - \psi(\xi)|}{|y-\xi|^\alpha}, \quad \frac{|\eta(y) - \eta(\xi)|}{|y-\xi|^\gamma} \tag{6.19}$$

are uniformly bounded for all $x, y \in \Gamma$, $\tau \in [-c,c]$, and $\xi = \vartheta_\tau(x)$ such that $y \neq \xi$. Combining (6.19) with (4.7) and (3.12), we obtain the decomposition

$$K_W[\eta,\psi](\xi,y) = \frac{A(\xi,y)}{|y-\xi|^{3-\alpha-\gamma}} + \frac{B(\xi,y)}{|y-\xi|^{2-\gamma}}, \tag{6.20}$$

where $A(\xi, y)$ and $B(\xi, y)$ are uniformly bounded functions defined for all $x, y \in \Gamma$, $\tau \in [-c, c]$, and $\xi = \vartheta_\tau(x)$ such that $y \neq \xi$. From this decomposition with $\tau = 0$ ($\xi = x$), together with the conditions $\gamma > 0$ and $\alpha + \gamma > 1$, we see that $K_W[\eta, \psi](x, y)$ is weakly singular for $x, y \in \Gamma$ and hence $\overline{U}(y)$ is finite for all $y \in \Gamma$.

Next, let $g(y, \tau) = U(y, \tau) - \overline{U}(y)$ and consider any $a \in (0, d_*]$ as in Lemma 6.1. For any $y \in \Gamma$ we have $\Gamma = (\Gamma \backslash \Gamma_{y,a}) \cup \Gamma_{y,a}$ and hence

$$\begin{aligned} g(y, \tau) = &\int_{\Gamma \backslash \Gamma_{y,a}} \Big[K_W[\eta, \psi](\vartheta_\tau(x), y) J_\tau(x) - K_W[\eta, \psi](x, y) \Big] \, dA_x \\ &+ \int_{\Gamma_{y,a}} K_W[\eta, \psi](\vartheta_\tau(x), y) J_\tau(x) \, dA_x - \int_{\Gamma_{y,a}} K_W[\eta, \psi](x, y) \, dA_x. \end{aligned} \tag{6.21}$$

For the second integral in (6.21) we have, using (6.4), (6.7), and (6.20), and the boundedness of the Jacobian and Lemma 6.1,

$$\begin{aligned} &\left| \int_{\Gamma_{y,a}} K_W[\eta, \psi](\vartheta_\tau(x), y) J_\tau(x) \, dA_x \right| \\ &\qquad \leq \int_{\Gamma_{y,a}} \frac{C}{|y - \vartheta_\tau(x)|^{3-\alpha-\gamma}} + \frac{C}{|y - \vartheta_\tau(x)|^{2-\gamma}} \, dA_x \\ &\qquad \leq \int_{\Gamma_{y,a}} \frac{C}{(r^2 + \tau^2)^{(3-\alpha-\gamma)/2}} + \frac{C}{(r^2 + \tau^2)^{(2-\gamma)/2}} \, dA_x \\ &\qquad \leq C a^{\alpha+\gamma-1} + C a^\gamma, \end{aligned} \tag{6.22}$$

where C is a constant independent of x, y, and τ. A bound of exactly the same form also holds for the third integral in (6.21). Moreover, from (6.6), we notice that the Jacobian can be written in the form $J_\tau(x) = 1 + \tau\omega(x, \tau)$, where $\omega(x, \tau)$ is uniformly bounded. Using the above results in (6.21), we obtain the following for all $y \in \Gamma$, $\tau \in [-c, c] \backslash \{0\}$, and $a \in (0, d_*]$:

$$\begin{aligned} |g(y, \tau)| \leq &\int_{\Gamma \backslash \Gamma_{y,a}} \Big| K_W[\eta, \psi](\vartheta_\tau(x), y) - K_W[\eta, \psi](x, y) \Big| \, dA_x \\ &+ \int_{\Gamma \backslash \Gamma_{y,a}} \Big| \tau K_W[\eta, \psi](\vartheta_\tau(x), y) \omega(x, \tau) \Big| \, dA_x + C a^{\alpha+\gamma-1} + C a^\gamma. \end{aligned} \tag{6.23}$$

To establish the limit relation for $g(y, \tau)$, let $\varepsilon > 0$ be given and fix $a \in (0, d_*]$ such that $Ca^{\alpha+\gamma-1} + Ca^\gamma \leq \varepsilon/3$. Then since the function $K_W[\eta, \psi](\vartheta_\tau(x), y) - K_W[\eta, \psi](x, y)$ is uniformly continuous on the compact set defined by $x, y \in \Gamma$, $|y - x| \geq a$, $\tau \in [-c, c]$, and this function vanishes when $\tau = 0$, and moreover the function $K_W[\eta, \psi](\vartheta_\tau(x), y)\omega(x, \tau)$ is uniformly bounded on the above compact set, we find there exists a $\delta > 0$ such that

$$\left. \begin{aligned} \Big| K_W[\eta, \psi](\vartheta_\tau(x), y) - K_W[\eta, \psi](x, y) \Big| &\leq \varepsilon/(3|\Gamma|) \\ \Big| \tau K_W[\eta, \psi](\vartheta_\tau(x), y) \omega(x, \tau) \Big| &\leq \varepsilon/(3|\Gamma|) \end{aligned} \right. \quad \left\{ \begin{array}{ll} \forall x, y \in \Gamma, & |y - x| \geq a, \\ \forall \tau \in [-c, c], & |\tau| \leq \delta. \end{array} \right. \tag{6.24}$$

Using (6.24) in (6.23) we find that $|g(y, \tau)| \leq \varepsilon$ for all $y \in \Gamma$ whenever $0 < |\tau| \leq \delta$, which shows that $g(y, \tau) \to 0$ as $\tau \to 0^\pm$ uniformly in y everywhere on Γ.

The announced property for the sum $M^\tau[\eta,\psi]+N^\tau[\eta,\psi]$ follows from the observation that, for any $\tau\in[-c,c]\backslash\{0\}$, we have, in view of Lemma 6.2,

$$M^\tau[\eta,\psi]+N^\tau[\eta,\psi]=2\int_\Gamma \overline{U}(y)+g(y,\tau)\ dA_y, \tag{6.25}$$

and hence

$$\begin{aligned}&\lim_{\tau\to 0^\pm}\Big[M^\tau[\eta,\psi]+N^\tau[\eta,\psi]\Big]\\ &\qquad=2\int_\Gamma \overline{U}(y)\ dA_y=2\int_\Gamma\int_\Gamma K_W[\eta,\psi](x,y)\ dA_x dA_y,\end{aligned} \tag{6.26}$$

which is the desired result. ∎

The next result shows that the one-sided limits of the difference $M^\tau[\eta,\psi]-N^\tau[\eta,\psi]$, as $\tau\to 0^\pm$, both vanish under a stricter Hölder condition on the function ψ. The proof of this result for the difference is significantly more involved and delicate than that for the sum and suggests that this stricter Hölder condition may be rather sharp.

LEMMA 6.4. *Let $\Gamma\in C^{1,1}$, $\psi\in C^{0,1}(\Gamma,\mathbb{R}^3)$, and $\eta\in C^{0,\gamma}(\Gamma,\mathbb{R}^3)$ be given, where $\gamma>0$. Then*

$$\lim_{\tau\to 0^\pm}\Big[M^\tau[\eta,\psi]-N^\tau[\eta,\psi]\Big]=0. \tag{6.27}$$

Proof. From (6.5) and (6.6), together with (6.10) and (6.11), and the fact that η and ν are constant in the normal direction, we obtain

$$\begin{aligned}&M^\tau[\eta,\psi]-N^\tau[\eta,\psi]\\ &\quad=\int_{\Gamma_\tau}\eta_i(\xi)\Sigma_W^{ik}[\Gamma,\psi](\xi)\nu_k(\xi)\ dA_\xi-\int_\Gamma \eta_i(y)\Sigma_W^{ik}[\Gamma_\tau,\psi](y)\nu_k(y)\ dA_y\\ &\quad=\int_\Gamma \eta_i(y)g^i(y,\tau)\ dA_y,\end{aligned} \tag{6.28}$$

where for any $y\in\Gamma$ and $\tau\in[-c,c]\backslash\{0\}$ we have

$$g^i(y,\tau)=\Sigma_W^{ik}[\Gamma,\psi](\vartheta_\tau(y))\nu_k(y)J_\tau(y)-\Sigma_W^{ik}[\Gamma_\tau,\psi](y)\nu_k(y). \tag{6.29}$$

To establish the announced property for the difference $M^\tau[\eta,\psi]-N^\tau[\eta,\psi]$, we seek to show that $g(y,\tau)\to 0$ as $\tau\to 0^\pm$ for almost every y on Γ and moreover that $g(y,\tau)$ is bounded by an integrable function on Γ, so that the dominated convergence theorem may be applied.

To study the limiting behavior of $g(y,\tau)$, we first rewrite (6.29) in a more useful form. To this end, we use (3.10) and the fact that the two bracketed integrals in (6.15) vanish, together with symmetry properties of Ξ_W^{ikjl} implied by (3.12), and the fact that ψ and ν are constant in the normal direction, to get

$$\begin{aligned}\Sigma_W^{ik}[\Gamma,\psi](\vartheta_\tau(y))&=\int_\Gamma \Xi_W^{ikjl}(\vartheta_\tau(y),x)\psi_j(x)\nu_l(x)\ dA_x\\ &=\int_\Gamma \Xi_W^{ikjl}(\vartheta_\tau(y),x)\widehat{\psi}_j(x,y)\nu_l(x)\ dA_x,\end{aligned} \tag{6.30}$$

and similarly,

$$(6.31)\qquad \begin{aligned} \Sigma_W^{ik}[\Gamma_\tau,\psi](y) &= \int_{\Gamma_\tau} \Xi_W^{ikjl}(y,\xi)\psi_j(\xi)\nu_l(\xi)\, dA_\xi \\ &= \int_{\Gamma_\tau} \Xi_W^{ikjl}(y,\xi)\widehat{\psi}_j(\xi,y)\nu_l(\xi)\, dA_\xi \\ &= \int_{\Gamma} \Xi_W^{ikjl}(y,\vartheta_\tau(x))\widehat{\psi}_j(x,y)\nu_l(x)J_\tau(x)\, dA_x. \end{aligned}$$

Substituting (6.30) and (6.31) into (6.29) we get

$$(6.32)\qquad \begin{aligned} &g^i(y,\tau) \\ &\quad = \int_\Gamma \left[\Xi_W^{ikjl}(\vartheta_\tau(y),x)J_\tau(y) - \Xi_W^{ikjl}(y,\vartheta_\tau(x))J_\tau(x)\right]\widehat{\psi}_j(x,y)\nu_l(x)\nu_k(y)\, dA_x. \end{aligned}$$

It will be convenient to decompose the integral in (6.32) into a telescoping-type sum, which will be helpful in isolating delicate terms. To this end, let d_* be the constant in Lemma 6.1 and consider any fixed $b \in (0, d_*]$. For any $y \in \Gamma$, let $\Gamma_{y,b}$ be the Lyapunov patch of radius b at y, and consider a local polar coordinate map for $\Gamma_{y,b}\backslash\{y\}$ of the form $x = \varphi_y^{\mathrm{polar}}(\rho,\theta)$ with inverse $(\rho,\theta) = \varphi_y^{\mathrm{polar},-1}(x)$, which for brevity we denote by $x_{\rho,\theta}$ and (ρ_x,θ_x). As in the proof of Lemma 6.3, we write the Jacobian in (6.6) in the form $J_\tau(x) = 1 + \tau\omega(x,\tau)$, where $\omega(x,\tau)$ is uniformly bounded. With these considerations in mind, we decompose the integral in (6.32) as

$$(6.33)\qquad g(y,\tau) = \mathcal{S}_A(y,\tau) + \mathcal{S}_B(y,\tau) + \mathcal{S}_C(y,\tau) + \mathcal{S}_D(y,\tau) + \mathcal{S}_E(y,\tau),$$

where for any $y \in \Gamma$ and $\tau \in [-c,c]\backslash\{0\}$ we have

$$(6.34)\qquad \mathcal{S}_A(y,\tau) = \int_\Gamma [G_1(x,y,\tau) - G_2(x,y,\tau)] - [F_1(x,y,\tau) - F_2(x,y,\tau)]\, dA_x,$$

$$(6.35)\qquad \mathcal{S}_B(y,\tau) = \int_{\Gamma\backslash\Gamma_{y,b}} [F_1(x,y,\tau) - F_2(x,y,\tau)]\, dA_x,$$

$$(6.36)\qquad \mathcal{S}_C(y,\tau) = \int_{\Gamma_{y,b}} [F_1(x,y,\tau) - F_2(x,y,\tau)] - F_3(x,y,\tau)\, dA_x,$$

$$(6.37)\qquad \mathcal{S}_D(y,\tau) = \int_{\Gamma_{y,b}} F_3(x,y,\tau)\, dA_x,$$

$$(6.38)\qquad \mathcal{S}_E(y,\tau) = \int_\Gamma \tau[G_1(x,y,\tau)\omega(y,\tau) - G_2(x,y,\tau)\omega(x,\tau)]\, dA_x.$$

In the above, $G_1(x,y,\tau)$ and $G_2(x,y,\tau)$ are functions arising directly in (6.32) after substituting for the two Jacobians, namely,

$$(6.39)\qquad \begin{aligned} G_1^i(x,y,\tau) &= \Xi_W^{ikjl}(\vartheta_\tau(y),x)\widehat{\psi}_j(x,y)\nu_l(x)\nu_k(y), \\ G_2^i(x,y,\tau) &= \Xi_W^{ikjl}(y,\vartheta_\tau(x))\widehat{\psi}_j(x,y)\nu_l(x)\nu_k(y), \end{aligned}$$

$F_1(x,y,\tau)$ and $F_2(x,y,\tau)$ are auxiliary functions introduced to simplify the analysis of $G_1(x,y,\tau)$ and $G_2(x,y,\tau)$, specifically

$$(6.40)\qquad \begin{aligned} F_1^i(x,y,\tau) &= H^{ikjl}(\vartheta_\tau(y),x)\widehat{\psi}_j(x,y)\nu_l(x)\nu_k(y), \\ F_2^i(x,y,\tau) &= H^{ikjl}(y,\vartheta_\tau(x))\widehat{\psi}_j(x,y)\nu_l(x)\nu_k(y), \end{aligned}$$

where $H^{ikjl}(y,x) = 6\delta_{lk} z_i z_j / r^5 - 30 z_i z_j z_k z_l / r^7$, $z = x - y$, $r = |z|$, and $F_3(x,y,\tau)$ is an auxiliary function introduced to simplify the analysis of $F_1(x,y,\tau)$ and $F_2(x,y,\tau)$, specifically

$$F_3^i(x,y,\tau) = \frac{(r^2+\tau^2)^{7/2}}{|x-\vartheta_\tau(y)|^7 |\vartheta_\tau(x)-y|^7} (48\tau^3 r^2 - 12\tau r^4) F_0^i(\theta_x, y), \tag{6.41}$$

where $F_0(\theta,y)$ and a related function $F_\rho(\theta,y)$ are defined in $\Gamma_{y,b}\backslash\{y\}$ as

$$\begin{aligned} F_0^i(\theta,y) &= \limsup_{\rho\downarrow 0} F_\rho^i(\theta,y), \\ F_\rho^i(\theta,y) &= \frac{\widehat{\psi}_j(x_{\rho,\theta},y)}{|x_{\rho,\theta}-y|} \left[\frac{(x_{\rho,\theta}-y)_i\ \nu_j(y)}{|x_{\rho,\theta}-y|} + \frac{(x_{\rho,\theta}-y)_j\ \nu_i(y)}{|x_{\rho,\theta}-y|} \right]. \end{aligned} \tag{6.42}$$

We next show that each of the integrals in (6.34)–(6.38) vanishes as $\tau \to 0^\pm$ either uniformly for every y on Γ or else for almost every y while being uniformly bounded by an integrable function on Γ. For the integral $\mathcal{S}_A(y,\tau)$ in (6.34), we use (3.12), (6.39), and (6.40), together with (6.5) and (6.7), to arrive at the bound for all $y \in \Gamma$, $x \in \Gamma_{y,b}\backslash\{y\}$ and $\tau \in [-c,c]\backslash\{0\}$,

$$\Big| [G_1(x,y,\tau) - G_2(x,y,\tau)] - [F_1(x,y,\tau) - F_2(x,y,\tau)] \Big| \le \frac{C\, r\, |\widehat{\psi}(x,y)|}{|x-\vartheta_\tau(y)|^3}, \tag{6.43}$$

where C is a constant independent of x, y, and τ, and we note that $|\widehat{\psi}(x,y)| \le C|x-y|^\alpha$ since $\psi \in C^{0,\alpha}$ on Γ; by assumption $\alpha = 1$, but any $\alpha > 0$ would suffice at this stage. Next, for any $y \in \Gamma$ and $a \in (0,b]$ consider the decomposition $\Gamma = (\Gamma\backslash\Gamma_{y,a}) \cup \Gamma_{y,a}$. Using this decomposition in (6.34), and employing (6.43) along with (6.7) and Lemma 6.1 for the portion over $\Gamma_{y,a}$, we get

$$\begin{aligned} &|\mathcal{S}_A(y,\tau)| \\ &\quad \le \int_{\Gamma\backslash\Gamma_{y,a}} \Big| [G_1(x,y,\tau) - G_2(x,y,\tau)] - [F_1(x,y,\tau) - F_2(x,y,\tau)] \Big|\, dA_x + Ca^\alpha. \end{aligned} \tag{6.44}$$

To establish the limit result, let $\varepsilon > 0$ be given and fix $a \in (0,b]$ such that $Ca^\alpha \le \varepsilon/2$. Then since the integrand in (6.44) is uniformly continuous on the compact set defined by $x,y \in \Gamma$, $|y-x| \ge a$, $\tau \in [-c,c]$, and this integrand vanishes when $\tau = 0$, we find there exists a $\delta > 0$ such that $|\mathcal{S}_A(y,\tau)| \le \varepsilon$ for all $y \in \Gamma$ whenever $0 < |\tau| \le \delta$, which shows that $\mathcal{S}_A(y,\tau) \to 0$ as $\tau \to 0^\pm$ uniformly in y everywhere on Γ. We remark that the role of the auxiliary functions $F_1(x,y,\tau)$ and $F_2(x,y,\tau)$ in (6.34) is to cancel corresponding terms in $G_1(x,y,\tau)$ and $G_2(x,y,\tau)$. This cancellation leads to the factor of r in the numerator in (6.43), which together with the Hölder estimate on ψ gives a bound that is only weakly singular on Γ, which leads to the limit result. Without the auxiliary functions, the bound in (6.43) would be more singular.

For convenience, we next consider the integrals $\mathcal{S}_B(y,\tau)$ and $\mathcal{S}_E(y,\tau)$ in (6.35) and (6.38). For $\mathcal{S}_B(y,\tau)$ we notice that, since $b \in (0,d_*]$ is fixed, the integrand is uniformly continuous on the compact set defined by $x,y \in \Gamma$, $|y-x| \ge b$, $\tau \in [-c,c]$, and this integrand vanishes when $\tau = 0$. Hence as above we find that $\mathcal{S}_B(y,\tau) \to 0$ as $\tau \to 0^\pm$ uniformly in y everywhere on Γ. For $\mathcal{S}_E(y,\tau)$, we use (3.12) and (6.39) to arrive at the following bounds for all $y \in \Gamma$, $x \in \Gamma_{y,b}\backslash\{y\}$ and $\tau \in [-c,c]\backslash\{0\}$:

$$\Big|\tau G_1(x,y,\tau)\omega(y,\tau)\Big|, \quad \Big|\tau G_2(x,y,\tau)\omega(x,\tau)\Big| \quad \le \quad \frac{C|\tau||\widehat{\psi}(x,y)|}{|x-\vartheta_\tau(y)|^3}, \tag{6.45}$$

where C is a constant independent of x, y, and τ, and also $|\widehat{\psi}(x,y)| \le C|x-y|^\alpha$ since $\psi \in C^{0,\alpha}$ on Γ. For any $y \in \Gamma$ and $a \in (0,b]$, we consider the decomposition $\Gamma = (\Gamma \backslash \Gamma_{y,a}) \cup \Gamma_{y,a}$ and argue as above using (6.45), (6.7), and Lemma 6.1 for integrals over $\Gamma_{y,a}$, and using the boundedness of $G_1(x,y,\tau)\omega(y,\tau)$ and $G_2(x,y,\tau)\omega(x,\tau)$ for integrals over $\Gamma\backslash\Gamma_{y,a}$, to conclude that $\mathcal{S}_E(y,\tau) \to 0$ as $\tau \to 0^\pm$ uniformly in y everywhere on Γ.

For the integral $\mathcal{S}_C(y,\tau)$ in (6.36), we use (6.40), (6.41), and (6.42) to arrive at the following decomposition for all $y \in \Gamma$, $x \in \Gamma_{y,b}\backslash\{y\}$, and $\tau \in [-c,c]\backslash\{0\}$:

(6.46)
$$\begin{aligned} &F_1(x,y,\tau) - F_2(x,y,\tau) - F_3(x,y,\tau) \\ &\quad = \frac{B(x,y,\tau)\, r\, |\widehat{\psi}(x,y)|}{|x-\vartheta_\tau(y)|^3} + \frac{(r^2+\tau^2)^{7/2}(48\tau^3 r^2 - 12\tau r^4)}{|x-\vartheta_\tau(y)|^7|\vartheta_\tau(x)-y|^7}[F_{\rho_x}(\theta_x,y) - F_0(\theta_x,y)], \end{aligned}$$

where $B(x,y,\tau)$ is a function that is bounded uniformly in x, y, and τ. Next, for $y \in \Gamma$ and $x \in \Gamma_{y,b}\backslash\{y\}$ let $\Delta_y(x) = F_{\rho_x}(\theta_x,y) - F_0(\theta_x,y)$. Notice that $\Delta_y(x)$ is bounded uniformly in x and y since $\psi \in C^{0,1}$. Moreover, for almost every y, $\Delta_y(x)$ is continuous in x and satisfies $\lim_{|x-y|\downarrow 0} \Delta_y(x) = 0$ due to the almost everywhere differentiability of $\psi \in C^{0,1}$. For any $\tau \in [-c,c]\backslash\{0\}$ and $a \in (0,b]$, we consider the decomposition $\Gamma_{y,b} = (\Gamma_{y,b}\backslash\Gamma_{y,a}) \cup \Gamma_{y,a}$, and with the aid of (6.7), and the facts that $r \le (r^2+\tau^2)^{1/2}$ and $|\tau| \le (r^2+\tau^2)^{1/2}$, and Lemma 6.1, we obtain

$$\begin{aligned} &\left| \int_{\Gamma_{y,a}} \frac{(r^2+\tau^2)^{7/2}(48\tau^3 r^2 - 12\tau r^4)\Delta_y(x)}{|x-\vartheta_\tau(y)|^7|\vartheta_\tau(x)-y|^7}\, dA_x \right| \\ &\qquad \le C \sup_{0<|x-y|<a} |\Delta_y(x)| \int_{\Gamma_{y,a}} \frac{|\tau|^3 r^2 + |\tau| r^4}{(r^2+\tau^2)^{7/2}}\, dA_x \\ &\qquad \le C \sup_{0<|x-y|<a} |\Delta_y(x)| \int_{\Gamma_{y,a}} \frac{|\tau|}{(r^2+\tau^2)^{3/2}}\, dA_x \\ &\qquad \le C \sup_{0<|x-y|<a} |\Delta_y(x)|. \end{aligned} \tag{6.47}$$

Moreover, since B is bounded and $\psi \in C^{0,1}$, we find, again using (6.7) and Lemma 6.1,

$$\left| \int_{\Gamma_{y,a}} \frac{B(x,y,\tau)\, r\, |\widehat{\psi}(x,y)|}{|x-\vartheta_\tau(y)|^3}\, dA_x \right| \le \int_{\Gamma_{y,a}} \frac{Cr^2}{(r^2+\tau^2)^{3/2}}\, dA_x \le Ca. \tag{6.48}$$

Applying the decomposition $\Gamma_{y,b} = (\Gamma_{y,b}\backslash\Gamma_{y,a}) \cup \Gamma_{y,a}$ to the integral in (6.36), and using (6.46), (6.47), and (6.48) for the portion over $\Gamma_{y,a}$, we get the following bound for all $y \in \Gamma$ and $\tau \in [-c,c]\backslash\{0\}$, where $0 < a \le b$ are fixed:

$$\begin{aligned} |\mathcal{S}_C(y,\tau)| \le &\int_{\Gamma_{y,b}\backslash\Gamma_{y,a}} |F_1(x,y,\tau) - F_2(x,y,\tau) - F_3(x,y,\tau)|\, dA_x \\ &+ C \sup_{0<|x-y|<a} |\Delta_y(x)| + Ca. \end{aligned} \tag{6.49}$$

To establish a limit result, we notice first that $\mathcal{S}_C(y,\tau)$ is uniformly bounded. Next, for almost every y, we recall that $\Delta_y(x)$ is continuous in x and satisfies $\lim_{|x-y|\downarrow 0} \Delta_y(x) = 0$, and hence for any given $\varepsilon > 0$ we can choose $a \in (0,b]$, depending on y, such that

$C\sup_{0<|x-y|<a}|\Delta_y(x)| + Ca \leq \varepsilon/2$. Then, since $F_1(x,y,\tau) - F_2(x,y,\tau)$ is uniformly continuous on the compact set defined by $x, y \in \Gamma$, $a \leq |y-x| \leq b$, $\tau \in [-c,c]$, and vanishes when $\tau = 0$, and $|F_3(x,y,\tau)| \leq C|\tau|$ on this compact set, we find there exists a $\delta > 0$ such that $|\mathcal{S}_C(y,\tau)| \leq \varepsilon$ whenever $0 < |\tau| \leq \delta$, which shows that $\mathcal{S}_C(y,\tau) \to 0$ as $\tau \to 0^{\pm}$ for almost every y. Hence, by the dominated convergence theorem, we conclude that $\mathcal{S}_C(y,\tau) \to 0$ in the L_1-norm on Γ as $\tau \to 0^{\pm}$.

The final integral we consider is $\mathcal{S}_D(y,\tau)$ in (6.37). In this case, we use (6.41) and (6.42), together with (6.4) and (6.7), to arrive at the following decomposition for all $y \in \Gamma$, $x \in \Gamma_{y,b}\backslash\{y\}$, and $\tau \in [-c,c]\backslash\{0\}$:

$$F_3(x,y,\tau) = \left[\frac{(48\tau^3\rho_x^2 - 12\tau\rho_x^4)}{(\rho_x^2+\tau^2)^{7/2}} + \frac{D(x,y,\tau)}{(\rho_x^2+\tau^2)^{1/2}}\right] F_0(\theta_x, y), \tag{6.50}$$

where (ρ_x, θ_x) are the polar coordinates of x, and $D(x,y,\tau)$ is a function that is bounded uniformly in x, y, and τ. Moreover, by (6.42) and the fact that $\psi \in C^{0,1}$, the function $F_0(\theta_x, y)$ is also bounded uniformly in x and y. For sufficiently small $a > 0$, we note that $\Gamma^0_{y,a} = \{x \in \Gamma \mid 0 \leq \rho_x \leq a,\ 0 \leq \theta_x < 2\pi\}$ is contained in $\Gamma_{y,b}$ and we consider the decomposition $\Gamma_{y,b} = (\Gamma_{y,b}\backslash\Gamma^0_{y,a}) \cup \Gamma^0_{y,a}$. Using the bound $1 \leq j_y(\zeta_x) \leq 1 + C\rho_x$ from (6.4), we obtain

$$\begin{aligned}
&\left|\int_{\Gamma^0_{y,a}} F_3(x,y,\tau)\, dA_x\right| \\
&\quad\leq \left|\int_0^{2\pi}\int_0^a \frac{(48\tau^3\rho^2 - 12\tau\rho^4)\rho}{(\rho^2+\tau^2)^{7/2}} F_0(\theta,y)\, d\rho\, d\theta\right| + \int_0^{2\pi}\int_0^a \frac{C\rho}{(\rho^2+\tau^2)^{1/2}}\, d\rho\, d\theta \\
&\quad\leq \left|\left[\int_0^{2\pi} F_0(\theta,y)\, d\theta\right]\left[\int_0^a \frac{(48\tau^3\rho^2 - 12\tau\rho^4)\rho}{(\rho^2+\tau^2)^{7/2}}\, d\rho\right]\right| + Ca \\
&\quad\leq \left|\left[\int_0^{2\pi} F_0(\theta,y)\, d\theta\right]\left[\frac{12a^4\tau}{(a^2+\tau^2)^{5/2}}\right]\right| + Ca \\
&\quad\leq \frac{Ca^4|\tau|}{(a^2+\tau^2)^{5/2}} + Ca.
\end{aligned} \tag{6.51}$$

Applying the decomposition $\Gamma_{y,b} = (\Gamma_{y,b}\backslash\Gamma^0_{y,a}) \cup \Gamma^0_{y,a}$ to the integral in (6.37), and using (6.51) for the portion over $\Gamma^0_{y,a}$, we get the following bound for all $y \in \Gamma$ and $\tau \in [-c,c]\backslash\{0\}$:

$$|\mathcal{S}_D(y,\tau)| \leq \int_{\Gamma_{y,b}\backslash\Gamma^0_{y,a}} |F_3(x,y,\tau)|\, dA_x + \frac{Ca^4|\tau|}{(a^2+\tau^2)^{5/2}} + Ca. \tag{6.52}$$

To establish the limit result, let $\varepsilon > 0$ be given and fix $a \in (0,b]$ such that $Ca \leq \varepsilon/3$. Then, in view of (6.41) we have $|F_3(x,y,\tau)| \leq C|\tau|$ for all $y \in \Gamma$ and $x \in \Gamma_{y,b}\backslash\Gamma^0_{y,a}$, and we find that there exists a $\delta > 0$ such that $|\mathcal{S}_D(y,\tau)| \leq \varepsilon$ for all $y \in \Gamma$ whenever $0 < |\tau| \leq \delta$. Hence $\mathcal{S}_D(y,\tau) \to 0$ as $\tau \to 0^{\pm}$ uniformly in y everywhere on Γ.

The announced property for the difference $M^\tau[\eta,\psi] - N^\tau[\eta,\psi]$ follows from (6.28) and (6.33). Specifically, the above results show that $g(y,\tau) \to 0$ in the L_1-norm on Γ as $\tau \to 0^{\pm}$; all terms in (6.33) converge in the C^0-norm, with the exception of $\mathcal{S}_C(y,\tau)$, which was only shown to converge in the L_1-norm. Hence, by the boundedness of η, we get

$$\begin{aligned}(6.53)\quad &\lim_{\tau\to 0^\pm}\Big[M^\tau[\eta,\psi]-N^\tau[\eta,\psi]\Big]\\ &\quad=\lim_{\tau\to 0^\pm}\int_\Gamma \eta_i(y)g^i(y,\tau)\,dA_y=\int_\Gamma \lim_{\tau\to 0^\pm}[\eta_i(y)g^i(y,\tau)]\,dA_y=0,\end{aligned}$$

which is the desired result. □

6.3. Proof of main result. Here we combine the results in Lemmas 6.3 and 6.4 to establish our main results in Theorem 4.1. Specifically, let $\Gamma\in C^{1,1}$, $\psi\in C^{0,1}(\Gamma,\mathbb{R}^3)$, and $\eta\in C^{0,\gamma}(\Gamma,\mathbb{R}^3)$ be given, where $\gamma>0$. For any $\tau\in[-c,c]\backslash\{0\}$ we have

$$(6.54)\qquad 2M^\tau[\eta,\psi]=\Big[M^\tau[\eta,\psi]+N^\tau[\eta,\psi]\Big]+\Big[M^\tau[\eta,\psi]-N^\tau[\eta,\psi]\Big].$$

In view of Lemmas 6.3 and 6.4, we see that the moment $M^\tau[\eta,\psi]$ has one-sided limits as $\tau\to 0^\pm$, and moreover

$$(6.55)\qquad \lim_{\tau\to 0^\pm} M^\tau[\eta,\psi]=\int_\Gamma\int_\Gamma K_W[\eta,\psi](x,y)\,dA_y dA_x.$$

The results in Theorem 4.1 for the double-layer traction moment $\mathcal{L}_W^\tau$ follow from the observation that $\mathcal{L}_W^\tau=M^\tau[\eta,\psi]$. The results for the single-layer traction moment $\mathcal{L}_V^\tau$ are straightforward consequences of the limit relations in (3.14) and (3.15) and the uniformity of these limits in the surface point. The details are omitted for brevity.

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