

## Chapter 2

# EUCLIDEAN PARALLEL POSTULATE

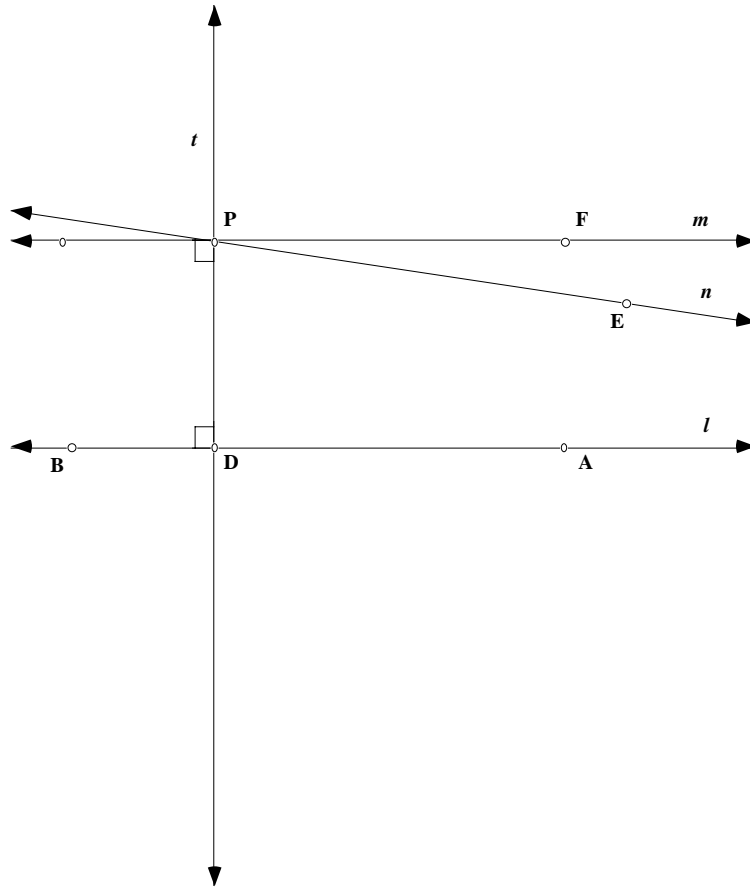
**2.1 INTRODUCTION.** There is a well-developed theory for a geometry based solely on the five Common Notions and first four Postulates of Euclid. In other words, there is a geometry in which neither the Fifth Postulate nor any of its alternatives is taken as an axiom. This geometry is called *Absolute Geometry*, and an account of it can be found in several textbooks - in Coxeter's book "Introduction to Geometry", for instance, - or in many college textbooks where the focus is on developing geometry within an axiomatic system. Because nothing is assumed about the existence or multiplicity of parallel lines, however, Absolute Geometry is not very interesting or rich. A geometry becomes a lot more interesting when some Parallel Postulate is added as an axiom! In this chapter we shall add the Euclidean Parallel Postulate to the five Common Notions and first four Postulates of Euclid and so build on the geometry of the Euclidean plane taught in high school. It is more instructive to begin with an axiom different from the Fifth Postulate.

**2.1.1 Playfair's Axiom.** Through a given point, not on a given line, exactly one line can be drawn parallel to the given line.

Playfair's Axiom is equivalent to the Fifth Postulate in the sense that it can be deduced from Euclid's five postulates and common notions, while, conversely, the Fifth Postulate can be deduced from Playfair's Axiom together with the common notions and first four postulates.

**2.1.2 Theorem.** Euclid's five Postulates and common notions imply Playfair's Axiom.

*Proof.* First it has to be shown that if  $P$  is a given point not on a given line  $l$ , then there is *at least one* line through  $P$  that is parallel to  $l$ . By Euclid's Proposition I 12, it is possible to draw a line  $t$  through  $P$  perpendicular to  $l$ . In the figure below let  $D$  be the intersection of  $l$  with  $t$ .



By Euclid's Proposition I 11, we can construct a line  $m$  through  $P$  perpendicular to  $t$ . Thus by construction  $t$  is a transversal to  $l$  and  $m$  such that the interior angles on the same side at  $P$  and  $D$  are both right angles. Thus  $m$  is parallel to  $l$  because the sum of the interior angles is  $180^\circ$ . (Note: Although we used the Fifth Postulate in the last statement of this proof, we could have used instead Euclid's Propositions I 27 and I 28. Since Euclid was able to prove the first 28 propositions without using his Fifth Postulate, it follows that the existence of *at least one* line through  $P$  that is parallel to  $l$ , can be deduced from the first four postulates. For a complete list of Euclid's propositions, see "College Geometry" by H. Eves, Appendix B.)

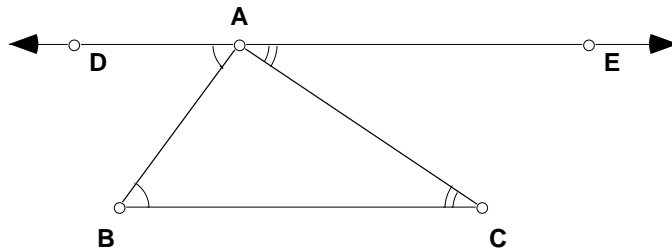
To complete the proof of 2.1.2, we have to show that  $m$  is the only line through  $P$  that is parallel to  $l$ . So let  $n$  be a line through  $P$  with  $m \neq n$  and let  $E \neq P$  be a point on  $n$ . Since  $m \neq n$ ,  $\angle EPD$  cannot be a right angle. If  $m\angle EPD < 90^\circ$ , as shown in the drawing, then  $m\angle EPD + m\angle PDA$  is less than  $180^\circ$ . Hence by Euclid's fifth postulate, the line  $n$  must intersect  $l$  on the same side of transversal  $t$  as  $E$ , and so  $n$  is not parallel to  $l$ . If  $m\angle EPD > 90^\circ$ , then a similar argument shows that  $n$  and  $l$  must intersect on the side of  $l$  opposite  $E$ . Thus,  $m$  is the one and only line through  $P$  that is parallel to  $l$ . **QED**

A proof that Playfair's axiom implies Euclid's fifth postulate can be found in most geometry texts. On page 219 of his "College Geometry" book, Eves lists eight axioms other than Playfair's axiom each of which is logically equivalent to Euclid's fifth Postulate, *i.e.*, to the Euclidean Parallel Postulate. A geometry based on the Common Notions, the first four Postulates and the Euclidean Parallel Postulate will thus be called *Euclidean* (plane) geometry. In the next chapter *Hyperbolic* (plane) geometry will be developed substituting Alternative B for the Euclidean Parallel Postulate (see text following Axiom 1.2.2)..

**2.2 SUM OF ANGLES.** One consequence of the Euclidean Parallel Postulate is the well-known fact that the sum of the interior angles of a triangle in Euclidean geometry is constant whatever the shape of the triangle.

**2.2.1 Theorem.** In Euclidean geometry the sum of the interior angles of any triangle is always  $180^\circ$ .

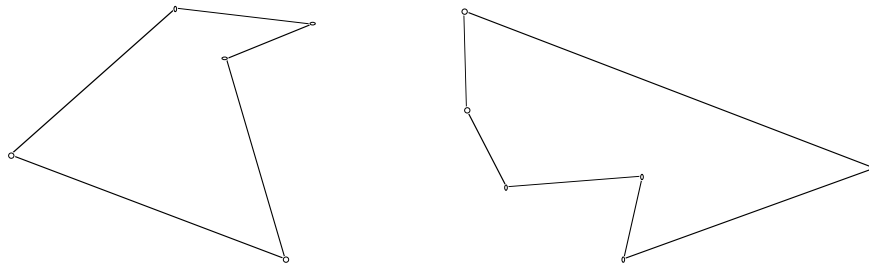
*Proof:* Let  $\triangle ABC$  be any triangle and construct the unique line  $\overleftrightarrow{DE}$  through A, parallel to the side  $\overline{BC}$ , as shown in the figure



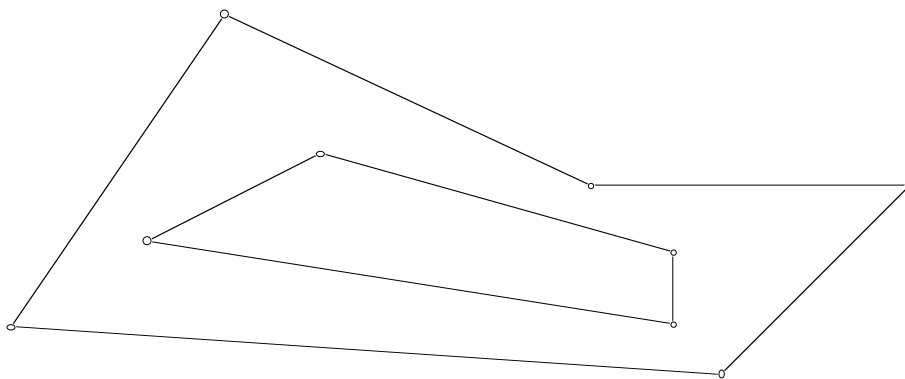
Then  $m\angle EAC = m\angle ACB$  and  $m\angle DAB = m\angle ABC$  by the alternate angles property of parallel lines, found in most geometry textbooks. Thus  $m\angle ACB + m\angle ABC + m\angle BAC = 180^\circ$ .

**QED**

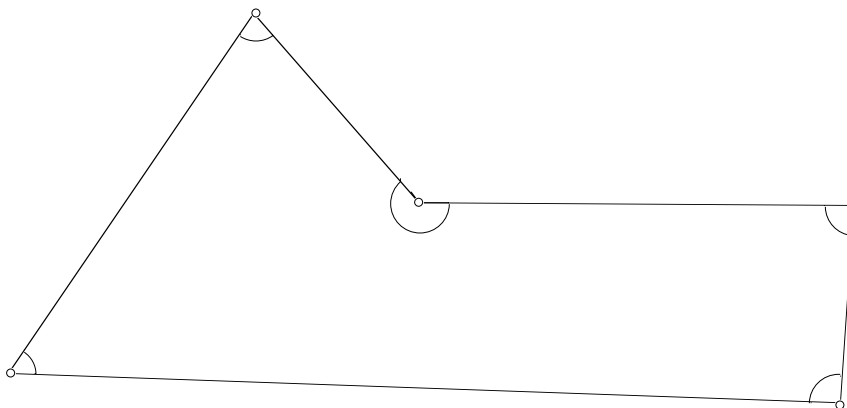
Equipped with Theorem 2.2.1 we can now try to determine the sum of the interior angles of figures in the Euclidean plane that are composed of a finite number of line segments, not just three line segments as in the case of a triangle. Recall that a *polygon* is a figure in the Euclidean plane consisting of points  $P_1, P_2, \dots, P_n$ , called *vertices*, together with line segments  $\overline{P_1P_2}, \overline{P_2P_3}, \dots, \overline{P_nP_1}$ , called *edges* or *sides*. More generally, a figure consisting of the union of a finite number of non-overlapping polygons will be said to be a *piecewise linear figure*. Thus



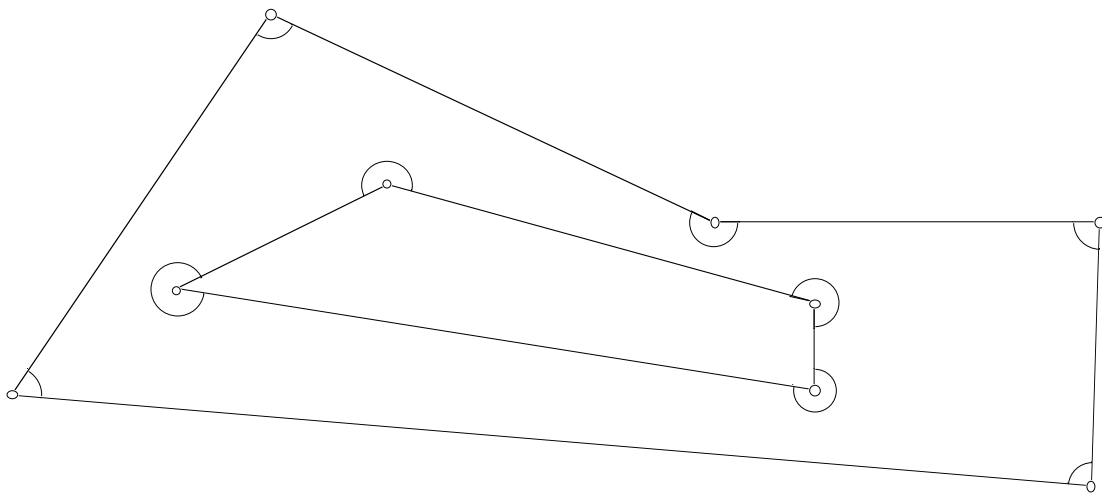
are piecewise linear figures as is the example of nested polygons below.



This example is a particularly interesting one because we can think of it as a figure containing a 'hole'. But is it clear what is meant by the interior angles of such figures? For such a polygon as the following:



we obviously mean the angles indicated. But what about a piecewise linear figure containing holes? For the example above of nested polygons, we shall mean the angles indicated below



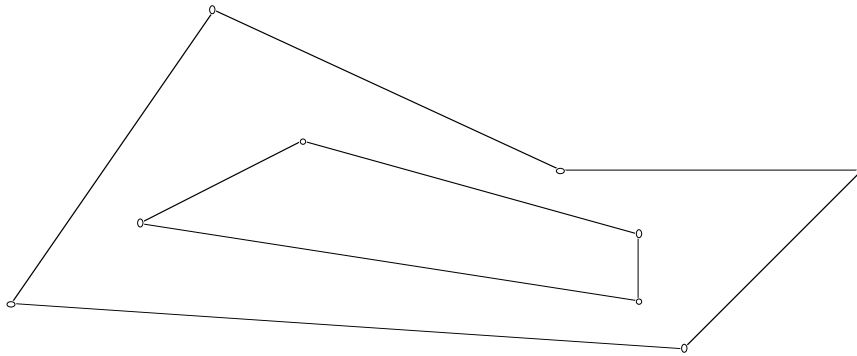
This makes sense because we are really thinking of the two polygons as enclosing a region so that *interior angle* then refers to the angle lying between two adjacent sides and inside the enclosed region. What this suggests is that for piecewise linear figures we will also need to specify what is meant by its *interior*.

The likely formula for the sum of the interior angles of piecewise linear figures can be obtained from Theorem 2.2.1 in conjunction with Sketchpad. In the case of polygons this was probably done in high school. For instance, the sum of the angles of any quadrilateral, *i.e.*, any four-sided figure, is  $360^\circ$ . To see this draw any diagonal of the quadrilateral thereby dividing the quadrilateral into two triangles. The sum of the angles of the quadrilateral is the sum of the angles of each of the two triangles and thus totals  $360^\circ$ . If the polygon has  $n$  sides, then it can be divided into  $n-2$  triangles and the sum of the angles of the polygon is equal to the sum of the angles of the  $n-2$  triangles. This proves the following result.

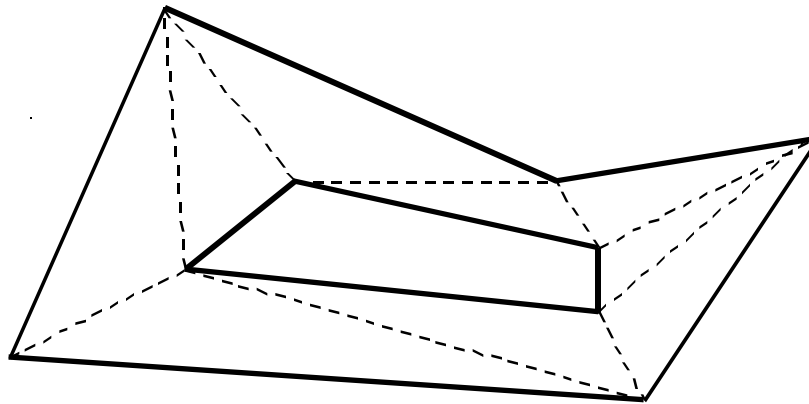
**2.2.2 Theorem.** The sum of the interior angles of an  $n$ -sided polygon,  $n \geq 3$ , is  $(n - 2) \cdot 180^\circ$ .

**2.2.2a Demonstration.**

We can use a similar method to determine the sum of the angles of the more complicated piecewise linear figures. One such figure is a polygon having “holes”, that is, a polygon having other non-overlapping polygons (the holes) contained totally within its interior. Open a new sketch and draw a figure such as



An interesting computer graphics problem is to color in the piecewise linear figure between the two polygons. Unfortunately, computer graphics programs will only fill polygons and the interior of the figure is not a polygon. Furthermore, Sketchpad measures angles greater than  $180^\circ$  by using directed measurements. Thus Sketchpad would give a measurement of  $-90^\circ$  for a  $270^\circ$  angle. To overcome the problem we use the same strategy as in the case of a polygon: *join enough of the vertices of the outer polygon to vertices on the inner polygon so that the region is sub-divided into polygons. Continue joining vertices until all of the polygons are triangles* as in the figure below. Color each of these triangles in a different color so that you can distinguish them easily.



We call this a *triangular tiling* of the figure. Now use Theorem 2.2.2 to compute the total sum of the angles of all these new polygons. Construct a different triangular tiling of the same figure and compute the total sum of angles again. Do you get the same value? Hence complete the following result.

**2.2.3 Theorem.** When an  $n$ -sided piecewise linear figure consists of a polygon with one polygonal hole inside it then the sum of its interior angles is \_\_\_\_\_.

Note: Here,  $n$  equals the number of sides of the outer polygon plus the number of sides of the polygonal hole.

**End of Demonstration 2.2.2a.**

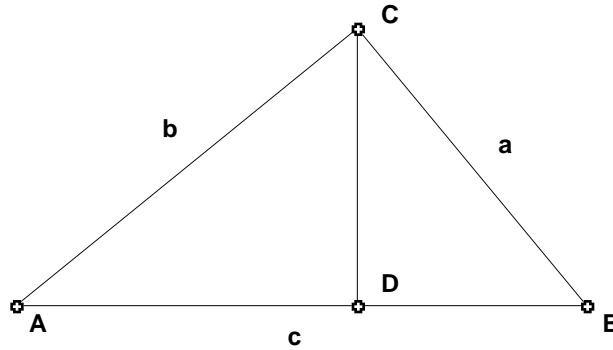
Try to prove Theorem 2.2.3 algebraically using Theorem 2.2.2. The case of a polygon containing  $h$  polygonal holes is discussed in Exercise 2.5.1.

## 2.3 SIMILARITY AND THE PYTHAGOREAN THEOREM

Of the many important applications of similarity, there are two that we shall need on many occasions in the future. The first is perhaps the best known of all results in Euclidean plane geometry, namely Pythagoras' theorem. This is frequently stated in purely algebraic terms as  $a^2 + b^2 = c^2$ , whereas in more geometrically descriptive terms it can be interpreted as saying that, *in area, the square built upon the hypotenuse of a right-angled triangle is equal to the sum of the squares built upon the other two sides*. There are many proofs of Pythagoras' theorem, some synthetic, some algebraic, and some visual as well as many combinations of these. Here you will discover an algebraic/synthetic proof based on the notion of similarity. Applications of Pythagoras' theorem and of its isosceles triangle version to decorative tilings of the plane will be made later in this chapter.

**2.3.4 Theorem. (The Pythagorean Theorem)** In any triangle containing a right angle, the square of the length of the side opposite to the right angle is equal to the sum of the squares of the lengths of the sides containing the right angle. In other words, if the length of the hypotenuse is  $c$  and the lengths of the other two sides are  $a$  and  $b$ , then  $a^2 + b^2 = c^2$ .

*Proof:* Let  $\triangle ABC$  be a right-angled triangle with right angle at  $C$ , and let  $\overline{CD}$  be the perpendicular from  $C$  to the hypotenuse  $\overline{AB}$  as shown in the diagram below.



- Show  $\triangle CAB$  is similar to  $\triangle DAC$ .
- Show  $\triangle CAB$  is similar to  $\triangle DCB$ .
- Now let  $\overline{BD}$  have length  $x$ , so that  $\overline{AD}$  has length  $c - x$ . By similar triangles,

$$\frac{x}{a} = \frac{a}{c} \quad \text{and} \quad \frac{c - x}{b} = ?$$

- Now eliminate  $x$  from the two equations to show  $a^2 + b^2 = c^2$ .

There is an important converse to the Pythagorean theorem that is often used.

**2.3.5 Theorem. (Pythagorean Converse)** Let  $\triangle ABC$  be a triangle such that  $a^2 + b^2 = c^2$ . Then  $\triangle ABC$  is right-angled with  $\angle ACB$  a right angle.

### 2.3.5a Demonstration (Pythagorean Theorem with Areas)

You may be familiar with the geometric interpretation of Pythagoras' theorem. If we build squares on each side of  $\triangle ABC$  then Pythagoras' theorem relates the area of the squares.

- Open a new sketch and draw a right-angled triangle  $\triangle ABC$ . Using the 'Square By Edge' tool construct an outward square on each edge of the triangle having the same edge length as the side of the triangle on which it is drawn.
- Measure the areas of these 3 squares: to do this select the vertices of a square and then construct its interior using "Construct Polygon Interior" tool. Now compute the area of each of these squares and then use the calculator to check that Pythagoras' theorem is valid for the right-angled triangle you have drawn.

**End of Demonstration 2.3.5a.**

This suggests a problem for further study because the squares on the three sides can be thought of as *similar* copies of the same piecewise linear figure with the lengths of the sides



determining the edge length of each copy. So what does Pythagoras' theorem become when the squares on each side are replaced by, say, equilateral triangles or regular pentagons? In order to investigate, we will need tools to construct other regular polygons given one edge. If you haven't already done so, move the document called `Polygons.gsp` into the Tool Folder and restart Sketchpad or simply open the document to make its tools available.

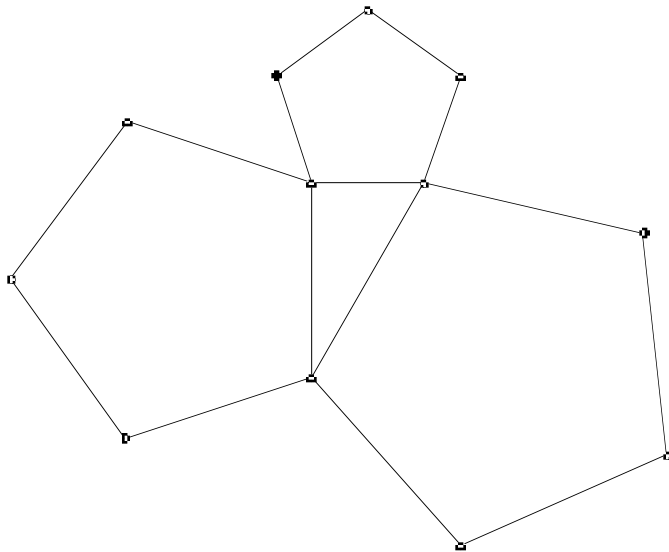
### 2.3.5b Demonstration (Generalization of Pythagorean Theorem)

- Draw a new right-angled triangle  $\triangle ABC$  and use the '5/Pentagon (By Edge)' script to construct an outward regular pentagon on each side having the same edge length as the side of the triangle on which it is drawn. As before measure the area of each pentagon. What do you notice about these areas?
- Repeat these constructions for an octagon instead of a pentagon. (Note: You can create an "Octagon By Edge" script from your construction for **Exercise 1.3.5(b).**) What do you notice about the areas in this case? Now complete the statement of **Theorem 2.3.6** below for regular  $n$ -gons.

**End of Demonstration 2.3.5b.**

**2.3.6 Theorem. (Generalization of Pythagoras' theorem)** When similar copies of a regular  $n$ -gon,  $n \geq 3$ , are constructed on the sides of a right-angled triangle, each  $n$ -gon having the same edge length as the side of the triangle on which it sits, then \_\_\_\_\_

The figure below illustrates the case of regular pentagons.



**2.3.7 Demonstration.** Reformulate the result corresponding to Theorem 2.3.6 when the regular  $n$ -gons constructed on each side of a right-angled triangle are replaced by similar triangles.

This demonstration presents an opportunity to explain another feature of Custom Tools called Auto-Matching. We will be using this feature in Chapter 3 when we use Sketchpad to explore the Poincaré Disk model of the hyperbolic plane. In this problem we can construct the first isosceles triangle and then we would like to construct two other similar copies of the original one. Here we will construct a “similar triangle script” based on the AA criteria for similarity.

### Tool Composition using Auto-Matching

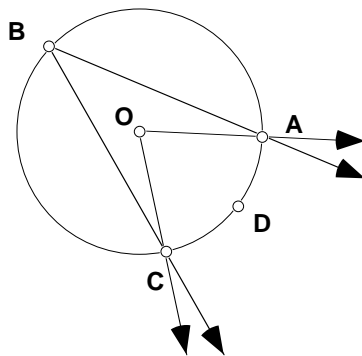
- Open a new sketch and construct  $\triangle ABC$  with the vertices labeled.
- Next construct the line (not a segment)  $\overline{DE}$ .
- Select the vertices  $B-A-C$  in order and choose "Mark Angle  $B-A-C$ " from the **Transform Menu**. Click the mouse to deselect those points and then select the point  $D$ . Choose “Mark Center  $D$ ” from the **Transform Menu**. Deselect the point and then select the line  $\overline{DE}$ . Choose “Rotate...” from the **Transform Menu** and then rotate by Angle  $B-A-C$ .
- Select the vertices  $A-B-C$  in order and choose “Mark Angle  $A-B-C$ ” from the **Transform Menu**. Click the mouse to deselect those points and then select the point  $E$ . Choose “Mark Center  $E$ ” from the **Transform Menu**. Deselect the point and then select the line  $\overline{DE}$ . Choose “Rotate...” from the **Transform Menu** and rotate by Angle  $A-B-C$ .
- Construct the point of intersection between the two rotated lines and label it  $F$ .  $\triangle DEF$  is similar to  $\triangle ABC$ . Hide the three lines connecting the points  $D$ ,  $E$ , and  $F$  and replace them with line segments.
- Now from the **Custom Tools** menu, choose **Create New Tool** and in the dialogue box, name your tool and check **Show Script View**. In the Script View, double click on the Given “Point  $A$ ” and a dialogue box will appear. Check the box labeled **Automatically Match Sketch Object**. Repeat the process for points  $B$  and  $C$ .

In the future, to use your tool, you need to have three points labeled  $A$ ,  $B$ , and  $C$  already constructed in your sketch where you want to construct the similar triangle. Then you only need to click on or construct the points corresponding to  $D$  and  $E$  each time you want to use the script. Your script will automatically match the points labeled  $A$ ,  $B$ , and  $C$  in your sketch with those that it needs to run the script. Notice in the Script View that the objects which are automatically matched are now listed under “Assuming” rather than under “Given Objects”. If there are no objects in the sketch with labels that match those in the Assuming section, then Sketchpad will require you to match those objects manually, as if they were “Given Objects.”

- Now open a new sketch and construct a triangle with vertices labeled  $A$ ,  $B$ , and  $C$ .
- In the same sketch, construct a right triangle. Use the “similar triangle” tool to build triangles similar to  $\triangle ABC$  on each side of the right triangle. For each similar triangle, select the three vertices and then in the Construct menu, choose “construct polygon interior”. Measure the areas of the similar triangles and see how they are related.

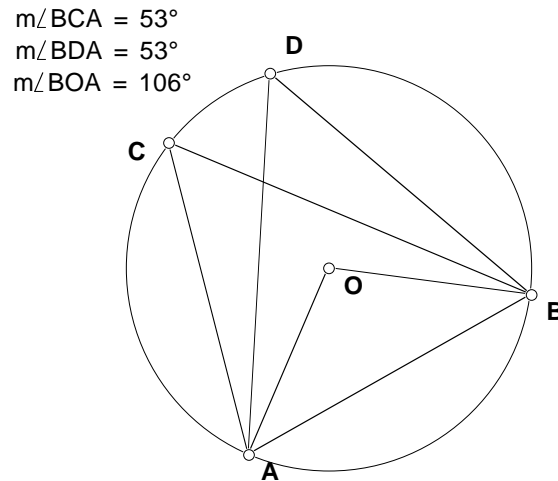
**End of Demonstration 2.3.7.**

**2.4 INSCRIBED ANGLE THEOREM:** One of the most useful properties of a circle is related to an angle that is inscribed in the circle and the corresponding subtended arc. In the figure below,  $\angle ABC$  is inscribed in the circle and  $Arc\ ADC$  is the subtended arc. We will say that  $\angle AOC$  is a central angle of the circle because the vertex is located at the center  $O$ . The measure of  $Arc\ ADC$  is defined to be the angle measure of the central angle,  $\angle AOC$ .



**2.4.0 Demonstration.** Investigate the relationship between an angle inscribed in a circle and the arc it intercepts (subtends) on the circle.

- Open a new script in *Sketchpad* and draw a circle, labeling the center of the circle by  $O$ .



- Select an arbitrary pair of points  $A, B$  on the circle. These points specify two possible arcs - let's choose the shorter one in the figure above, that is, the arc which is subtended by a central angle of measure less than  $180^\circ$ . Now select another pair of points  $C, D$  on the circle and draw line segments to form  $\angle BCA$  and  $\angle BDA$ . Measure these angles. What do you observe?
- If you drag  $C$  or  $D$  what do you observe about the angle measures? Now find the angle measure of  $\angle BOA$ . What do you observe about its value?
- Drag  $B$  until the line segment  $\overline{AB}$  passes through the center of the circle. What do you now observe about the three angle measures you have found?

Use your observations to complete the following statements; proving them will be part of later exercises.

**2.4.1 Theorem. (Inscribed Angle Theorem):** The measure of an inscribed angle of a circle equals \_\_\_\_\_ that of its intercepted (or subtended) arc.

**2.4.2 Corollary.** A diameter of a circle always inscribes \_\_\_\_\_ at any point on the circumference of the circle.

**2.4.3 Corollary.** Given a line segment  $\overline{AB}$ , the locus of a point  $P$  such that  $\angle APB = 90^\circ$  is a circle having  $\overline{AB}$  as diameter.

**End of Demonstration 2.4.0.**

The result you have discovered in Corollary 2.4.2 is a very useful one, especially in constructions, since it gives another way of constructing right-angled triangles. Exercises 2.5.4 and 2.5.5 below are good illustrations of this. The Inscribed Angle Theorem can also be used to prove the following theorem, which is useful for proving more advanced theorems.

**2.4.4 Theorem.** A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary. (A quadrilateral that is inscribed in a circle is called a *cyclic quadrilateral*.)

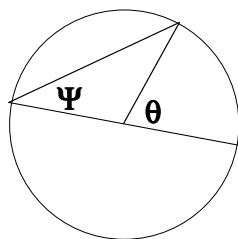
**2.5 Exercises**

**Exercise 2.5.1.** Consider a piecewise linear figure consisting of a polygon containing  $h$  holes (non-overlapping polygons in the interior of the outer polygon) has a total of  $n$  edges, where  $n$  includes both the interior and the exterior edges. Express the sum of the interior angles as a function of  $n$  and  $h$ . Prove your result is true.

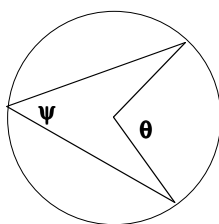
**Exercise 2.5.2.** Prove that if a quadrilateral is cyclic, then the opposite angles of the quadrilateral are supplementary, *i.e.*, the sum of opposite angles is  $180^\circ$ . [ This will provide half of the proof of Theorem 2.4.4. ]

**Exercise 2.5.3.** Give a synthetic proof of the Inscribed Angle Theorem 2.4.1 using the properties of isosceles triangles in Theorem 1.4.6. Hint: there are three cases to consider: here  $\psi$  is the angle subtended by the arc and  $\theta$  is the angle subtended at the center of the circle. The problem is to relate  $\psi$  to  $\theta$ .

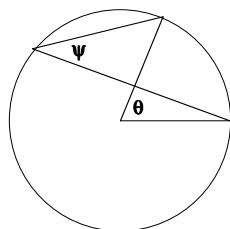
Case 1: The center of the circle lies on the subtended angle.



Case 2: The center of the circle lies within the interior of the inscribed circle.



Case 3: The center of the circle lies in the exterior of the inscribed angle.

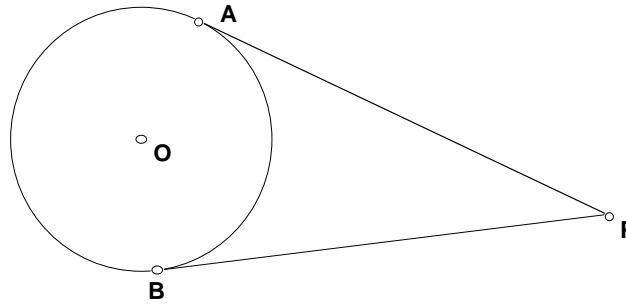


**End of Exercise 2.5.3.**

For Exercises 2.5.4, 2.5.5, and 2.5.6, recall that any line tangent to a circle at a particular point must be perpendicular to the line connecting the center and that same point. For all three of these exercises, the Inscribed Angle Theorem is useful.

**Exercise 2.5.4.** Use the Inscribed Angle Theorem to devise a Sketchpad construction that will construct the tangents to a given circle from a given point  $P$  outside the circle. Carry out your construction. (Hint: Remember Corollary 2.4.2).

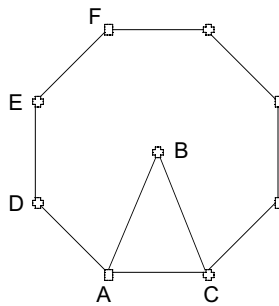
**Exercise 2.5.5.** In the following figure



the line segments  $\overline{PA}$  and  $\overline{PB}$  are the tangents to a circle centered at  $O$  from a point  $P$  outside the circle. Prove that  $\overline{PA}$  and  $\overline{PB}$  are congruent.

**Exercise 2.5.6.** Let  $l$  and  $m$  be lines intersecting at some point  $P$  and let  $Q$  be a point on  $l$ . Use the result of Exercise 2.5.5 to devise a Sketchpad construction that constructs a circle tangential to  $l$  and  $m$  that passes through  $Q$ . Carry out your construction.

For Exercises 2.5.7 and 2.5.8, we consider regular polygons again, that is, polygons with all sides congruent and all interior angles congruent. If a regular polygon has  $n$  sides we shall say it is a regular  $n$ -gon. For instance, the following figure



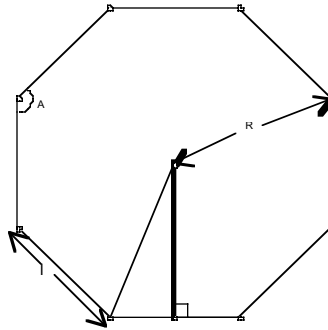
is a regular octagon above, *i.e.*, a regular 8-gon. By Theorem 2.2.2 the interior angle of a regular  $n$ -gon is

$$\left(\frac{n-2}{n}\right)180^\circ.$$

The measure of any central angle is  $\frac{360^\circ}{n}$ . In the figure  $\angle DEF$  is an interior angle and  $\angle ABC$  is a central angle.

**Exercise 2.5.7.** Prove that the vertices of a regular polygon always lie on a circumscribing circle. (Be careful! Don't assume that your polygon has a center; you must *prove* that there is a point equidistant from all the vertices of the regular polygon.)

**Exercise 2.5.8.** Now suppose that the edge length of a regular  $n$ -gon is  $l$  and let  $R$  be the *radius* of the circumscribing circle for the  $n$ -gon. The *Apothem* of the  $n$ -gon is the perpendicular distance from the center of the circumscribing circle to a side of the  $n$ -gon.



**The Apothem**

- (a) With this notation and terminology and using some trigonometry complete the following  
 $R = l$  \_\_\_\_\_ ,  $l = R$  \_\_\_\_\_ , *Apothem* =  $R$  \_\_\_\_\_ .

Use this to deduce

(b) area of  $n$ -gon =  $\frac{1}{2}nR^2 \sin\left(\frac{2\pi}{n}\right)$ , (c) perimeter of  $n$ -gon =  $2nR \sin\left(\frac{\pi}{n}\right)$ .

- (d) Then use the well-known fact from calculus that

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

to derive the formulas for the area of a circle of radius  $R$  as well as the circumference of such a circle.

**Exercise 2.5.9.** Use Exercise 2.5.8 together with the usual version of Pythagoras' theorem to give an algebraic proof of the generalized Pythagorean Theorem (Theorem 2.3.6).

**Exercise 2.5.10** Prove the converse to the Pythagorean Theorem stated in Theorem 2.3.5.

**2.6 RESULTS REVISITED.** In this section we will see how the Inscribed Angle Theorem can be used to prove results involving the Simson Line, the Miquel Point, and the

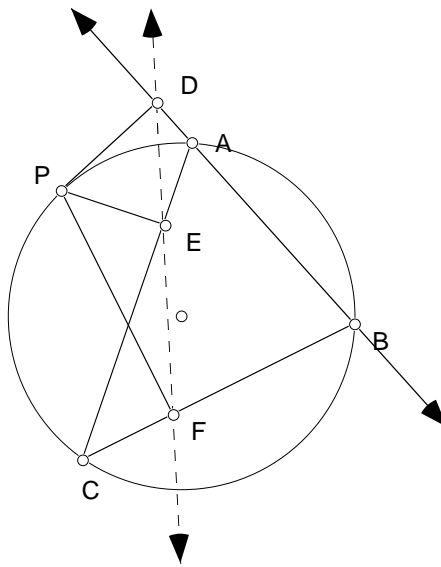


Euler Line. Recall that we discovered the Simson Line in Section 1.8 while exploring Pedal Triangles.

**2.6.1 Theorem (Simson Line).** If  $P$  lies on the circumcircle of  $\triangle ABC$ , then the perpendiculars from  $P$  to the three sides of the triangle intersect the sides in three collinear points.

*Proof.* Use the notation in the figure below.

- Why do  $P, D, A,$  and  $E$  all lie on the same circle? Why do  $P, A, C,$  and  $B$  all lie on another circle? Why do  $P, D, B$  and  $F$  all lie on a third circle? Verify all three of these statements using Sketchpad.



- In circle  $PDAE$ ,  $m\angle PDE \cong m\angle PAE = m\angle PAC$ . Why?
- In circle  $PACB$ ,  $m\angle PAC \cong m\angle PBC = m\angle PBF$ . Why?
- In circle  $PDBF$ ,  $m\angle PBF \cong m\angle PDF$ . Why?

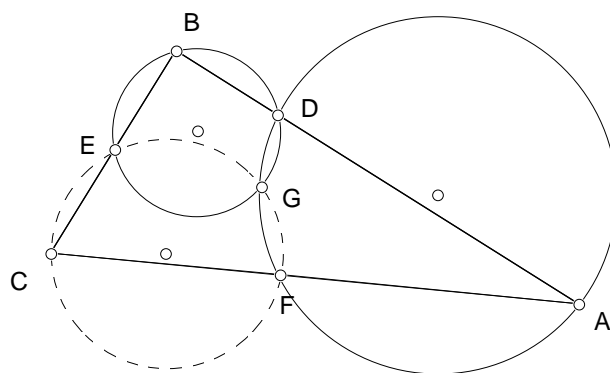
Since  $m\angle PDE \cong m\angle PDF$ , points  $D, E,$  and  $F$  must be collinear.

**QED**

In Exercise 1.9.4, the Miquel Points of a triangle were constructed.

**2.6.2 Theorem (Miquel Point)** If three points are chosen, one on each side of a triangle, then the three circles determined by a vertex and the two points on the adjacent sides meet at a point called the Miquel Point.

*Proof.* Refer to the notation in the figure below.



Let  $D$ ,  $E$  and  $F$  be arbitrary points on the sides of  $\triangle ABC$ . Construct the three circumcircles. Suppose the circumcircles for  $\triangle AFD$  and  $\triangle BDE$  intersect at point  $G$ . We need to show the third circumcircle also passes through  $G$ . Now,  $G$  may lie inside  $\triangle ABC$ , on  $\triangle ABC$ , or outside  $\triangle ABC$ . We prove the theorem here in the case that  $G$  lies inside  $\triangle ABC$ , and leave the other two cases for you (see Exercise 2.8.1).

- $\angle FGD$  and  $\angle DAF$  are supplementary. Why?
- $\angle EGD$  and  $\angle DBE$  are supplementary. Why?

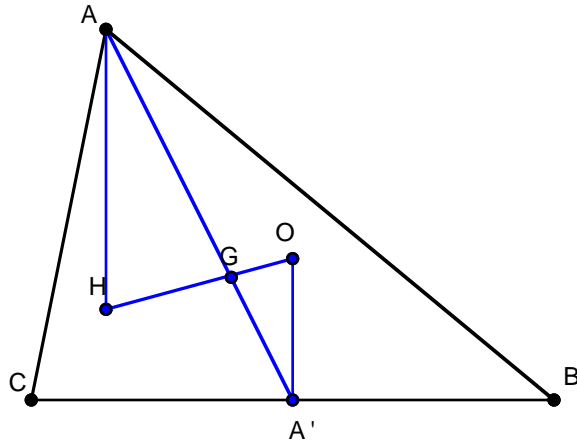
Notice  $m\angle FGD + m\angle DGE + m\angle EGF = 360^\circ$ . Combining these facts we see the following.  $(180^\circ - m\angle A) + (180^\circ - m\angle B) + m\angle EGF = 360^\circ$ . So  $m\angle EGF = 180 - m\angle C$  or  $\angle C$  and  $\angle EGF$  are supplementary. Thus  $C$ ,  $E$ ,  $G$ , and  $F$  all lie on a circle and the third circumcircle must pass through  $G$ . **QED**

The proof of Theorem 2.6.3 below uses two results on the geometry of triangles, which were proven in Chapter 1. The first result states that the line segment between the midpoints of two sides of a triangle is parallel to the third side of the triangle and it is half the length of the third side (see **Corollary 1.5.4**). The second result states that the point which is  $2/3$  the distance from a vertex (along a median) to the midpoint of the opposite side is the centroid of the triangle (see **Theorem 1.5.6**).

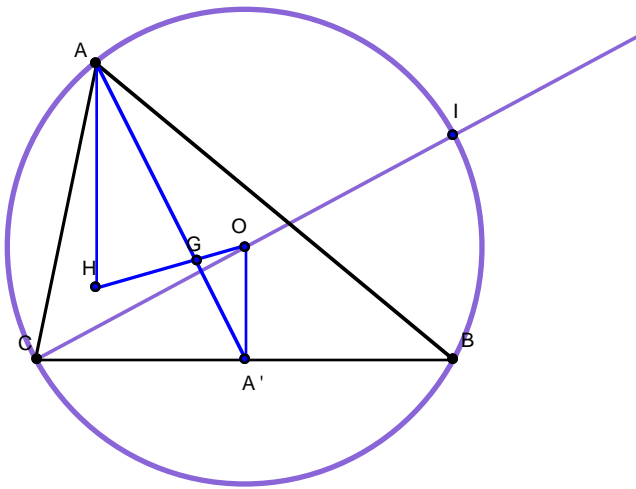
**2.6.3 Theorem (Euler Line).** For any triangle, the centroid, the orthocenter, and the circumcenter are collinear, and the centroid trisects the segment joining the orthocenter and the

circumcenter. The line containing the centroid, orthocenter, and circumcenter of a triangle is called the **Euler Line**.

**Proof.** In the diagram below,  $A'$  is the midpoint of the side opposite to  $A$  and  $O$ ,  $G$ , and  $H$  are the circumcenter, centroid, and orthocenter, respectively. Since  $A$ ,  $G$ , and  $A'$  are collinear, we can show that  $O$ ,  $G$ , and  $H$  are also collinear, by showing that  $\angle AGH \cong \angle A'GO$ . To do this, it suffices to show that  $\triangle AHG \sim \triangle A'OG$ . If we also show that the ratio of similiarity is 2:1, then we will also prove that  $G$  trisects  $\overline{OH}$ .



The proof that  $\triangle AHG \sim \triangle A'OG$  with ratio 2:1 proceeds as follows: Let  $I$  be the point where the ray  $\overline{CO}$  intersects the circumcircle of  $\triangle ABC$ . Then  $\overline{IB} \perp \overline{CB}$  (why?). It follows that  $\triangle BCI \sim \triangle A'CO$  with ratio 2:1 (why?) It is also true that  $AIBH$  is a parallelogram (why?) and



hence  $AH = IB = 2(OA')$ . Since  $G$  is the median, we know that  $AG = 2(GA')$ . Thus we have two corresponding sides proportional. The included angles are congruent because they are

alternate interior angles formed by the parallel lines  $\overline{AH}$  and  $\overline{OA'}$  and the transversal  $\overline{AA'}$ . (Why are  $\overline{AH}$  and  $\overline{OA'}$  parallel?) Thus,  $\triangle AHG \sim \triangle A'OG$  with ratio 2:1 by SAS.

Of course, as we noted in Chapter 1, we must be careful not to rely too much on a picture when proving a theorem. Use Sketchpad to find examples of triangles for which our proof breaks down, i.e. triangles in which we can't form the triangles  $\triangle AHG$  and  $\triangle A'OG$ . What sorts of triangles arise? You should find two special cases. Finish the proof of Theorem 2.6.3 by proving the result for each of these cases (see **Exercise 2.8.2**).

**2.7 THE NINE POINT CIRCLE.** Another surprising triangle property is the so-called Nine-Point Circle, sometimes credited to K.W. Feuerbach (1822). Sketchpad is particularly well adapted to its study. The following Demonstration will lead you to its discovery.

**2.7.0 Demonstration:** Investigate the nine points on the Nine Point Circle.

**The First set of Three points:**

- In a new sketch construct  $\triangle ABC$ . Construct the midpoints of each of its sides; label these midpoints  $D$ ,  $E$ , and  $F$ .
- Construct the circle that passes through  $D$ ,  $E$ , and  $F$ . (You know how to do this!)
- This circle is called the *Nine-Point Circle*. Complete the statement: The nine-point circle passes through \_\_\_\_\_.

**The Second set of Three points:** In general the nine-point circle will intersect  $\triangle ABC$  in three more points. If yours does not, drag one of the vertices around until the circle does intersect  $\triangle ABC$  in three other points. Label these points  $J$ ,  $K$ , and  $L$ .

- Construct the line segment joining  $J$  and the vertex opposite  $J$ . Change the color of this segment to red. What is the relationship between the red segment and the side of the triangle containing  $J$ ? What is an appropriate name for the red segment?
- Construct the corresponding segment joining  $K$  and the vertex opposite  $K$  and the segment joining  $L$  to the vertex opposite  $L$ . Color each segment red. What can you say about the three red segments?
- Place a point where the red segments meet; label this point  $M$  and complete the following statement: The nine-point circle also passes through \_\_\_\_\_.

**The Third set of Three points:** The red segments intersect the circle at their respective endpoints ( $J$ ,  $K$ , or  $L$ ). For each segment there exists a second point where the segment intersects the circle. Label them  $N$ ,  $O$  and  $P$ .

- To describe these points measure the distance between  $M$  and each of  $A$ ,  $B$ , and  $C$ . Measure also the distance between  $M$  and each of  $N$ ,  $O$ , and  $P$ . What do you observe? Confirm your observation by dragging the vertices of  $\triangle ABC$ .
- Complete the following statement: The nine-point circle also passes through \_\_\_\_\_

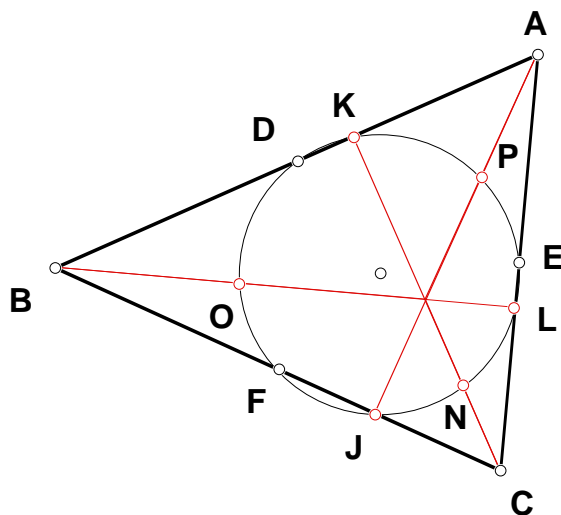
You should create a Nine Point Circle tool from this sketch and save it for future use.

### **End of Demonstration 2.7.0.**

To understand the proof of Theorem 2.7.1 below, it is helpful to recall some results discussed earlier. As in the proof of the existence of the Euler Line, it is necessary to use the fact that the segment connecting the midpoints of two sides of a triangle is parallel to the third side of the triangle. Also, we recall that a quadrilateral can be inscribed in a circle if and only if the opposite angles in the quadrilateral are supplementary. It is not difficult to show that an isosceles trapezoid has this property. Finally, recall that a triangle can be inscribed in a circle with a side of the triangle coinciding with a diameter of the circle if and only if the triangle is a right triangle.

**2.7.1 Theorem (The Nine-point Circle)** The midpoints of the sides of a triangle, the points of intersection of the altitudes and the sides, and the midpoints of the segments joining the orthocenter and the vertices all lie on a circle called the nine-point circle.

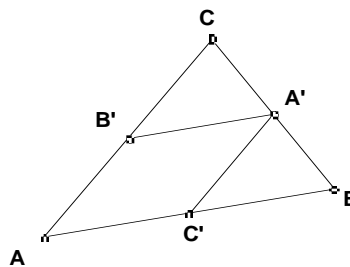
Your final figure should be similar to



*Proof:*

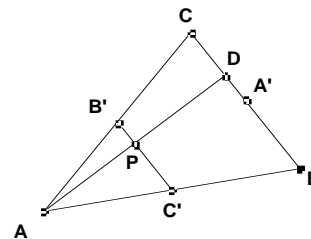
(See Figure 1) In  $\triangle ABC$  label the midpoints of  $\overline{BC}$ ,  $\overline{CA}$ , and  $\overline{AB}$ , by  $A'$ ,  $B'$  and  $C'$  respectively. There is a circle containing  $A'$ ,  $B'$  and  $C'$ . In addition, we know  $A'C'AB'$  is a parallelogram, and so  $A'C' = AB'$ .

Figure 1



(See Figure 2) Construct the altitude from  $A$  intersecting  $\overline{BC}$  at  $D$ . As  $\overline{A'B'C'}$  is parallel to  $\overline{BC}$  and  $\overline{AD}$  is perpendicular to  $\overline{BC}$ , then  $\overline{AD}$  must be perpendicular to  $\overline{A'B'C'}$ . Denote the intersection of  $\overline{A'B'C'}$  and  $\overline{AD}$  by  $P$ . Then  $\angle APB \cong \angle DPB$ ,  $\overline{PB} \cong \overline{PB}$  and  $\overline{AP} \cong \overline{DP}$ .

Figure 2



(See Figure 3) Consequently,  $\triangle APB \cong \triangle DPB$  by SAS. So  $AB' = B'D$ . By transitivity with  $A'C' = AB'$

Figure 3

$$A'C' = AB'$$

we have  $B'D = A'C'$ . Thus  $A'C'B'D$  is an isosceles trapezoid. Hence, by the remarks preceding this theorem,  $A', C', B'$ , and  $D$  are points which lie on one circle. (See Figure 4)

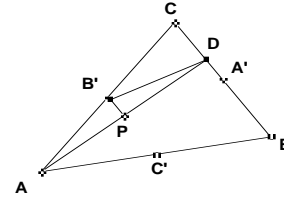


Figure 4

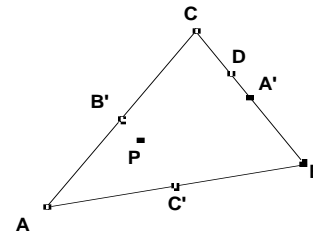


Figure 5

(See Figure 5) Let  $J$  denote the midpoint of the segment joining vertex  $A$  and the orthocenter  $H$ . Then, again by the connection of midpoints of the sides of a triangle,  $\overline{CJ}$  is parallel to  $\overline{BH}$ .

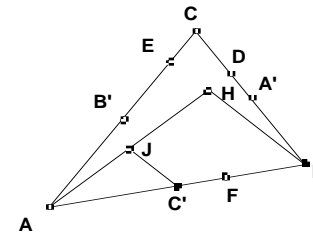


Figure 6

(See Figure 6) Now  $\overline{C'A} \perp \overline{AC}$  and  $\overline{AC} \perp \overline{BH}$  but  $\overline{BH} \parallel \overline{C'J}$ . Hence  $\overline{C'A} \perp \overline{C'J}$ .

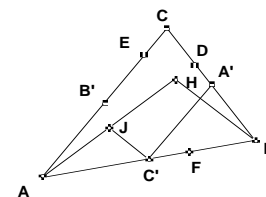
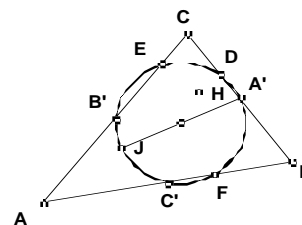


Figure 7

(See Figure 7) Therefore  $C'$  lies on a circle with diameter  $\overline{AJ}$ .

A similar argument shows that  $B'$  lies on the circle with diameter  $\overline{AC}$ , and hence  $J$  lies on the circle determined by  $A'$ ,  $B'$ , and  $C'$ . Likewise, the other two midpoints of the segments joining the vertices with the orthocenter lie on the same circle.



**QED**

**2.8 Exercises.** In this exercise set, Exercise 2.8.3 – 2.8.7 are related to the nine point circle.

**Exercise 2.8.1.** Using Sketchpad, illustrate a case where the Miquel Point lies outside the triangle. Prove Theorem 2.6.2 in this case.

**Exercise 2.8.2.** Prove Theorem 2.6.3 for the two special cases:

- (a) The triangle is isosceles.
- (b) The triangle is a right triangle.

**Exercise 2.8.3.** For special triangles some points of the nine-point circle coincide. Open a new sketch and draw an arbitrary  $\triangle ABC$ . Explore the various possibilities by dragging the vertices of  $\triangle ABC$ . Describe the type of triangle (if it exists) for which the nine points of the nine-point circle reduce to:

- 4 points: \_\_\_\_\_
- 6 points: \_\_\_\_\_
- 8 points: \_\_\_\_\_
- 5 points: \_\_\_\_\_
- 7 points: \_\_\_\_\_

**Exercise 2.8.4.** Open a new sketch and draw an arbitrary triangle  $\triangle ABC$ .

- Construct the circumcenter  $O$ , the centroid  $G$ , the orthocenter  $H$ , and the center of the nine-point circle  $N$  for this triangle. What do you notice?
- Measure the length of  $\overline{ON}$ ,  $\overline{NH}$ ,  $\overline{NG}$ , and  $\overline{OH}$ . What results for a general triangle do your calculations suggest?

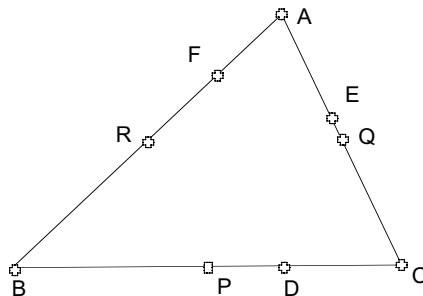


- Measure the radius of the nine-point circle of  $\triangle ABC$ . Measure the radius of the circumcircle of  $\triangle ABC$ . What results for a general triangle do your calculations suggest?

Drag the vertices of the triangle around. Do your conjectures still remain valid?

**Exercise 2.8.5.** Open a new sketch and draw an arbitrary  $\triangle ABC$ . Let  $H$  be the orthocenter and  $O$  be the circumcenter of  $\triangle ABC$ . Construct the nine-point circles for  $\triangle OHA$ ,  $\triangle OHB$ , and  $\triangle OHC$ . Use sketchpad to show that these nine-point circles have two points in common. Can you identify these points? Check your observation by dragging the vertices  $A$ ,  $B$ , and  $C$  around

If one starts with given vertices  $A$ ,  $B$ , and  $C$ , then the locations of the midpoints  $P$ ,  $Q$ , and  $R$  of the sides of  $\triangle ABC$  are uniquely determined. Similarly, the locations of the feet of the altitudes  $D$ ,  $E$ , and  $F$  will be determined once  $A$ ,  $B$ , and  $C$  are given. The remaining two problems in this exercise set use the geometric properties we have developed so far to reverse this process, *i.e.*, we construct the vertices  $A$ ,  $B$ , and  $C$  knowing the midpoints or the feet of the altitudes. Use the notation from the following figure.



**Exercise 2.8.6.**

- Prove the line segment  $\overline{PQ}$  is parallel to side  $\overline{AB}$ .
- Given points  $P$ ,  $Q$ , and  $R$ , show how to construct points  $A$ ,  $B$ , and  $C$  so that  $P$ ,  $Q$ , and  $R$  are the midpoints of the sides of  $\triangle ABC$ .
- Formulate a conjecture concerning the relation between the centroid  $G$  of  $\triangle ABC$  and the centroid of  $\triangle PQR$ .

**Exercise 2.8.7.**

(a) Assume  $\triangle ABC$  is acute (to ensure the feet of the altitudes lie on the sides of the triangle). Prove that  $PC = PB = PE = PF$  and that  $P$  lies on the perpendicular bisector of the line segment  $\overline{EF}$ .

(b) Given points  $D, E,$  and  $F,$  show how to construct points  $A, B,$  and  $C$  so that  $D, E,$  and  $F$  are the feet of the altitudes from the vertices of  $\triangle ABC$  to the opposite sides. (Hint: remember the nine-point circle).

**2.9 THE POWER OF A POINT AND SYNTHESIZING APOLLONIUS.**

Another application of similarity will be to a set of ideas involving what is often called the *power of a point with respect to a circle*. The principal result will be decidedly useful later in connection with the theory of inversion and its relation to hyperbolic geometry.

**Demonstration 2.9.0.** Discover the formula for *the power of point  $P$  with respect to a given circle*.

- Open a new sketch and draw a circle. Select any point  $P$  outside the circle and let  $A, B$  be the points of intersection on the circle of a line  $l$  through  $P$ .
- Compute the lengths  $PA, PB$  of  $\overline{PA},$  and  $\overline{PB}$  respectively; then compute the product  $PA \cdot PB$  of  $PA$  and  $PB$ . Drag  $l$  while keeping  $P$  fixed. What do you observe?
- Investigate further by considering the case when  $l$  is tangential to the given circle. Use this to explain your previous observation.
- What happens to the product  $PA \cdot PB$  when  $P$  is taken as a point on the circle?
- Now let  $P$  be a point inside the circle,  $l$  a line through  $P$  and  $A, B$  its points of intersection with the circle. Again compute the product  $PA \cdot PB$  of  $PA$  and  $PB$ . Now vary  $l$ .
- Investigate further by considering the case when  $l$  passes through the center of the given circle.
- Can you reconcile the three values of the product  $PA \cdot PB$  for  $P$  outside, on and inside the given circle? Hint: consider the value of  $OP^2 - r^2$  where  $O$  is the center of the given circle and  $r$  is its radius.

**End of Demonstration 2.9.0.**

The value of  $PA \cdot PB$  in Demonstration 2.9.0 is often called the *power of  $P$  with respect to the given circle*. Now complete the following statement.

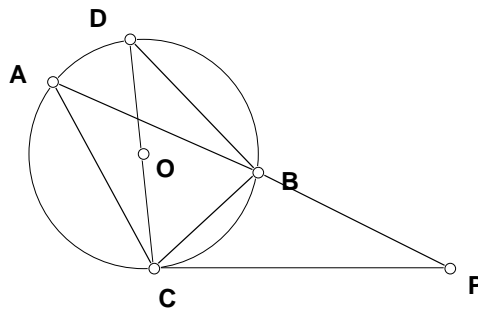
**2.9.1 Theorem.** Let  $P$  be a given point,  $\Sigma$  a given circle, and  $l$  a line through  $P$  intersecting  $\Sigma$  at  $A$  and  $B$ . Then

1. the product  $PA \cdot PB$  of the distances from  $P$  to  $A$  and  $B$  is \_\_\_\_\_ whenever  $P$  is outside, whenever it is inside or when it is on  $\Sigma$  ;
2. the value of the product  $PA \cdot PB$  is equal to \_\_\_\_\_ where  $O$  is the center of  $\Sigma$  and  $r$  is the radius of  $\Sigma$  .

The proof of part 2 of Theorem 2.9.1 in the case when  $P$  is outside the given circle is an interesting use of similarity and the inscribed angle theorem. In the diagram below let  $C$  be a point on the circle such that  $\overline{PC}$  is a tangent to the circle. By the Pythagorean Theorem  $OP^2 - r^2 = PC^2$  so it suffices to show that  $PA \cdot PB = PC^2$ .

**2.9.2 Theorem.** Given a circle  $\Sigma$  and a point  $P$  outside  $\Sigma$ , let  $l$  be a ray through  $P$  intersecting  $\Sigma$  at points  $A$  and  $B$ . If  $C$  is a point on the circle such that  $\overline{PC}$  is a tangent to  $\Sigma$  at  $C$  then  $PA \cdot PB = PC^2$ .

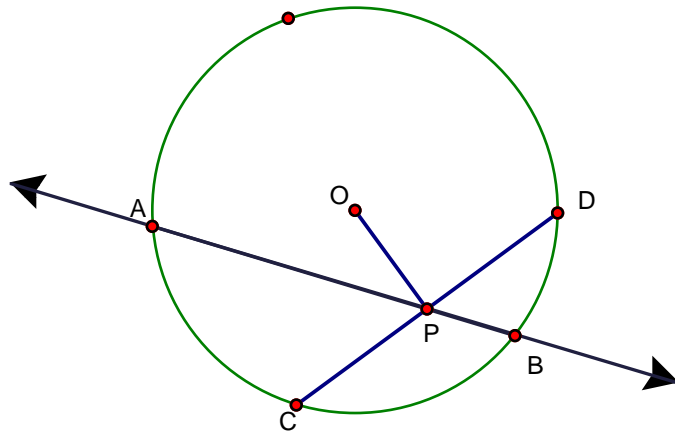
*Proof.* The equation  $PA \cdot PB = PC^2$  suggests use of similar triangles, but which ones?



Let  $\overline{CD}$  be a diameter of the circle. By the Inscribed Angle Theorem  $m\angle PAC = m\angle BDC$  and  $\angle CBD$  is a right angle. Thus  $m\angle BDC + m\angle DCB = 90^\circ$  and as  $\overline{PC}$  is tangent to the circle  $m\angle DCB + m\angle PCB = 90^\circ$ . Therefore,  $m\angle PAC = m\angle PCB$ . By AA similarity  $\triangle PAC$  is similar to  $\triangle PCB$  proving  $PA/PC = PC/PB$  or  $PA \cdot PB = PC^2$ . **QED**

**Theorem 2.9.3.** Given a circle  $\Sigma$  and a point  $P$  inside  $\Sigma$ , let  $l$  be a line through  $P$  intersecting  $\Sigma$  at points  $A$  and  $B$ . Let  $\overline{CD}$  be the chord perpendicular to the segment  $\overline{OP}$ . Then the value of the product  $PA \cdot PB$  is equal to  $r^2 - OP^2 = PC^2$  where  $O$  is the center of  $\Sigma$  and  $r$  is the radius of  $\Sigma$ .

*Proof.*



By AA similarity  $\triangle ACP$  is similar to  $\triangle DBP$  so that  $PA/PC = PD/PB$ . Thus  $PA \cdot PB = PC \cdot PD$ . By HL,  $\triangle CPO$  is congruent to  $\triangle DPO$  so that  $PC = PD$ . By the Pythagorean Theorem  $PD^2 + OP^2 = OD^2$ . Re-arranging and substituting, we obtain  $PC \cdot PD = r^2 - OP^2$ . Therefore,  $PA \cdot PB = r^2 - OP^2$  as desired. **QED**

There is a converse to theorem 2.9.2 that also will be useful later. You will be asked to provide the proof in Exercise 2.11.1 below.

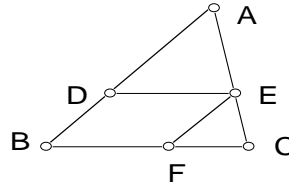
**2.9.4 Theorem.** Given a circle  $\Sigma$  and a point  $P$  outside  $\Sigma$ , let  $l$  be a ray through  $P$  intersecting  $\Sigma$  at points  $A$  and  $B$ . If  $C$  is a point on  $\Sigma$  such that  $PA \cdot PB = PC^2$ , then  $\overline{PC}$  is a tangent to  $\Sigma$  at  $C$ .

In Chapter 1 we used Sketchpad to discover that when a point  $P$  moves so that the distance from  $P$  to two fixed points  $A, B$  satisfies the condition  $PA = 2PB$  then the path traced out by  $P$  is a circle. In fact, the locus of a point  $P$  such that  $PA = mPB$  is always a circle, when  $m$  is any positive constant not equal to one. From restorations of Apollonius' work '*Plane Loci*' we infer that he considered this locus problem, now called the "Circle of Apollonius". However, this is a misnomer since Aristotle who had used it to give a mathematical justification of the semicircular form of the rainbow had already known the result.

That this locus is a circle was confirmed algebraically using coordinate geometry in Chapter 1. However, it can be also be proven by synthetic methods and the synthetic proof exploits properties of similar triangles and properties of circles. Since the synthetic proof will suggest how we can construct the *Circle of Apollonius* with respect to fixed points  $A, B$  through an arbitrary point  $P$  we shall go through the proof now. The proof requires several lemmas, which we consider below.

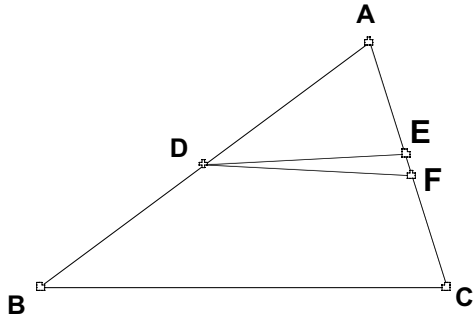
**2.9.5 Lemma** Given  $\triangle ABC$ , let  $D$  be on  $\overline{AB}$ , and  $E$  on  $\overline{AC}$  such that  $\overline{DE}$  is parallel to  $\overline{BC}$ . Then

$$\frac{AD}{DB} = \frac{AE}{EC} \text{ and } \frac{AB}{DB} = \frac{AC}{EC}.$$



*Proof.* Let  $F$  be the intersection of  $\overline{BC}$  with the line parallel to  $\overline{AB}$  passing through  $E$ . Then  $\triangle AED \sim \triangle ECF$  by AA similarity and  $\frac{AD}{EF} = \frac{AE}{EC}$ . The quadrilateral  $EFBD$  is a parallelogram, therefore  $EF = DB$  and  $\frac{AD}{DB} = \frac{AE}{EC}$ . A similar argument shows  $\frac{AB}{DB} = \frac{AC}{EC}$ . **QED**

**2.9.5a Lemma (Converse of Lemma 2.9.5).** Given  $\triangle ABC$ , let  $D$  be on  $\overline{AB}$ , and  $E$  on  $\overline{AC}$  such that  $\frac{AD}{DB} = \frac{AE}{EC}$  or  $\frac{AB}{DB} = \frac{AC}{EC}$  (see figure below), then  $\overline{DE}$  is parallel to  $\overline{BC}$ .



*Proof.* Assume  $\frac{AB}{DB} = \frac{AC}{EC}$ . The line through

$D$  parallel to  $\overline{BC}$  intersects  $\overline{AC}$  at point  $F$  with

$\overline{DF}$  parallel to  $\overline{BC}$ . By Lemma 2.9.5,

$\frac{AB}{DB} = \frac{AC}{FC}$ . But  $\frac{AB}{DB} = \frac{AC}{EC}$  also, so

$\frac{AC}{FC} = \frac{AC}{EC}$  which implies that  $F = E$ . Thus

$\overline{DE} = \overline{DF}$  is parallel to  $\overline{BC}$ .

If  $\frac{AD}{DB} = \frac{AE}{EC}$ , the proof is similar. **QED**

**2.9.6 Theorem** The bisector of the internal angle  $\angle ABC$  of  $\triangle ABC$  divides the opposite side  $\overline{AC}$  in the ratio of the adjacent sides  $\overline{BA}$  and  $\overline{BC}$ . In other words,

$$\frac{AD}{DC} = \frac{AB}{BC}.$$

*Proof.* Suppose  $\overline{BD}$  bisects  $\angle ABC$  in  $\triangle ABC$ . At  $C$  construct a line parallel to  $\overline{BD}$ , intersecting  $\overline{AB}$  at  $E$ , producing the figure below.

But then  $\angle ABD \cong \angle CBD$  and  $\angle BEC \cong \angle ABD$  since they are corresponding angles of parallel lines.

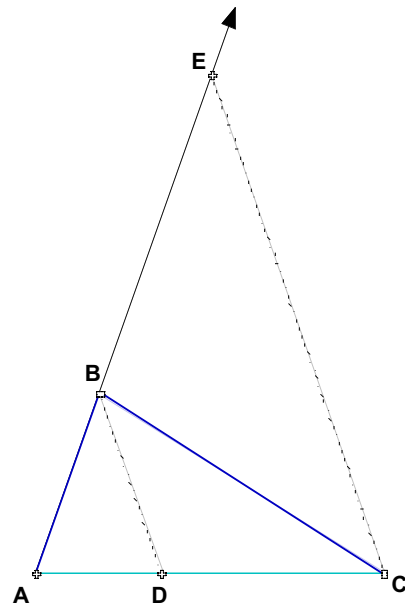
In addition,  $\angle BCE \cong \angle CBD$  since they are alternate interior angles of parallel lines. Hence

$\triangle CBE$  is isosceles and  $BE = BC$ . By the previous lemma

$$\frac{AB}{BE} = \frac{AD}{DC}.$$

But  $BE = BC$ , so

$$\frac{AB}{BC} = \frac{AD}{DC}.$$



This completes the proof.

**QED**

**2.9.7 Exercise.** The converse to Theorem 2.9.6 states that if

$$\frac{AB}{BC} = \frac{AD}{DC},$$

then  $\overline{BD}$  bisects  $\angle ABC$  in the figure above. Prove this converse. You may use the converse to Lemma 2.9.5, proven in Lemma 2.9.5a.

**2.9.8. Theorem** The bisector of an external angle of  $\triangle ABC$  cuts the extended opposite side at a point determined by the ratio of the adjacent sides. That is to say, if  $\overline{AB}$  is extended and intersects the line containing the bisector of the exterior angle of  $C$  at  $E$ , then

$$\frac{AC}{BC} = \frac{AE}{BE}.$$

**Proof:** There are two cases to consider.

Either  $m\angle BAC < m\angle ABC$  or  $m\angle BAC > m\angle ABC$ . (If  $m\angle BAC = m\angle ABC$ , then the bisector of the exterior angle at  $C$  is parallel to  $\overline{AB}$ .)

Assume that  $m\angle BAC < m\angle ABC$ . Then (as shown in the figure) the bisector of  $\angle BCG$  will intersect the extension of  $\overline{AB}$  at  $E$ , and  $AE > AB$ . At  $B$ , construct a line parallel to  $\overline{CE}$ , intersecting  $\overline{AC}$  at  $F$ .

Then  $\angle BFC \cong \angle ECG$   
since they are  
corresponding angles  
of parallel lines;

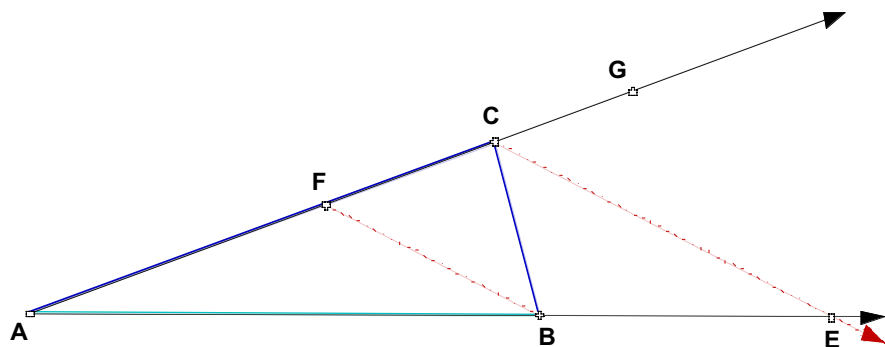
And  $\angle ECG \cong \angle BCE$   
since  $\overline{CE}$  bisects  
 $\angle BCG$ ; and

$\angle BCE \cong \angle CBF$  since they are alternate interior angles of parallel lines.

Hence  $\triangle BFC$  is isosceles and  $FC = BC$ . Now by a previous lemma,  $\frac{AC}{FC} = \frac{AE}{BE}$ . But

$$FC = BC; \text{ so } \frac{AC}{BC} = \frac{AE}{BE}.$$

This proves the assertion for the case when  $m\angle BAC < m\angle ABC$ .



If  $m\angle BAC > m\angle ABC$ , then the line containing the bisector of  $\angle BCG$  intersects the extension of  $\overline{AB}$  at point  $E$  on the other side of  $A$ , with  $A$  between  $E$  and  $B$ . A similar argument proves the assertion for this case as well and the theorem is proved. **QED**

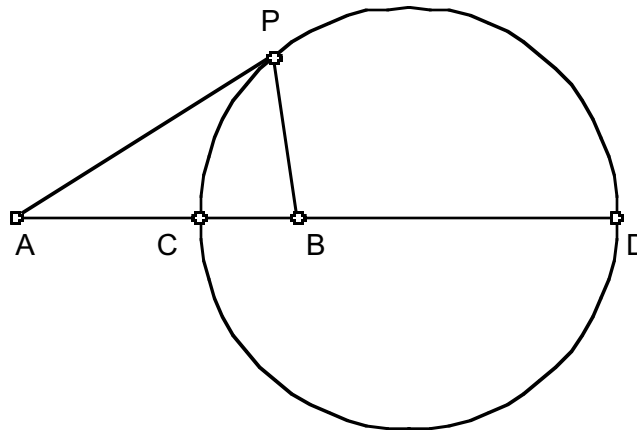
**2.9.9 Exercise.** The converse to Theorem 2.9.8 states that if

$$\frac{AC}{BC} = \frac{AE}{BE}$$

in the figure above, then  $\overline{CE}$  bisects the external angle of  $\triangle ABC$  at  $C$ . Prove this conjecture.

We are now able to complete the proof of the main theorem.

**2.9.10 Theorem (Circle of Apollonius).** The set of all points  $P$  such that the ratio of the distances to two fixed points  $A$  and  $B$  (that is  $PA/PB$ ) is constant (but not equal to 1) is a circle.

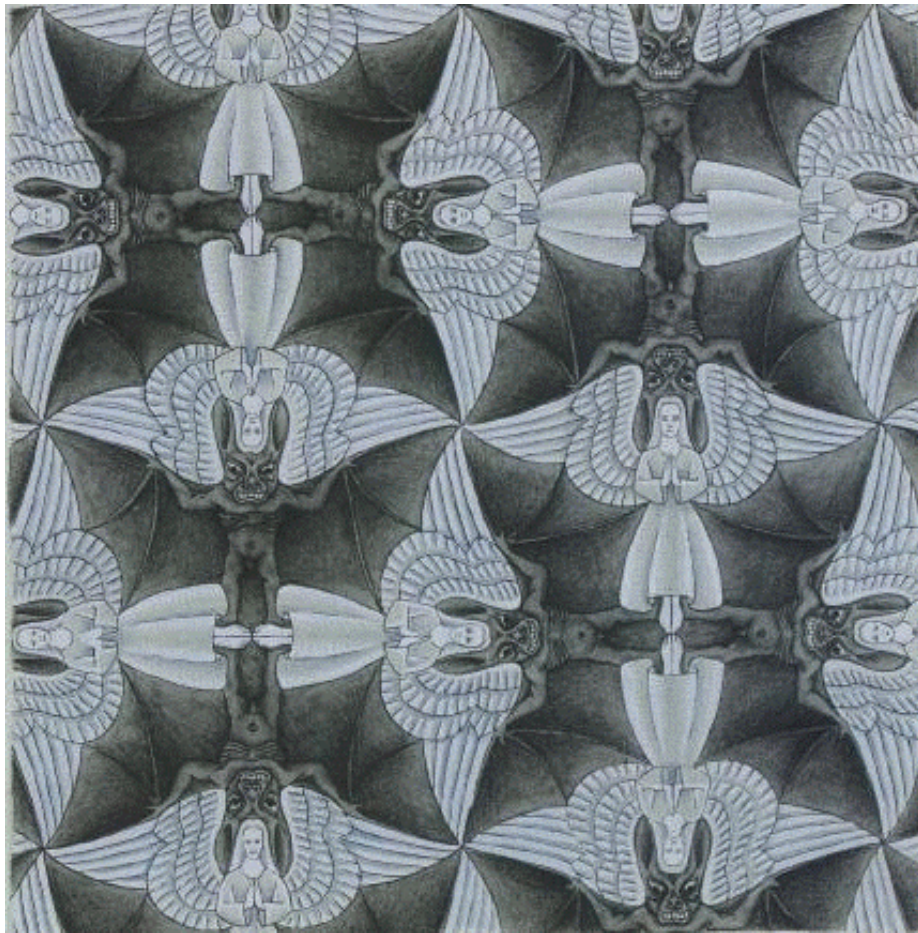


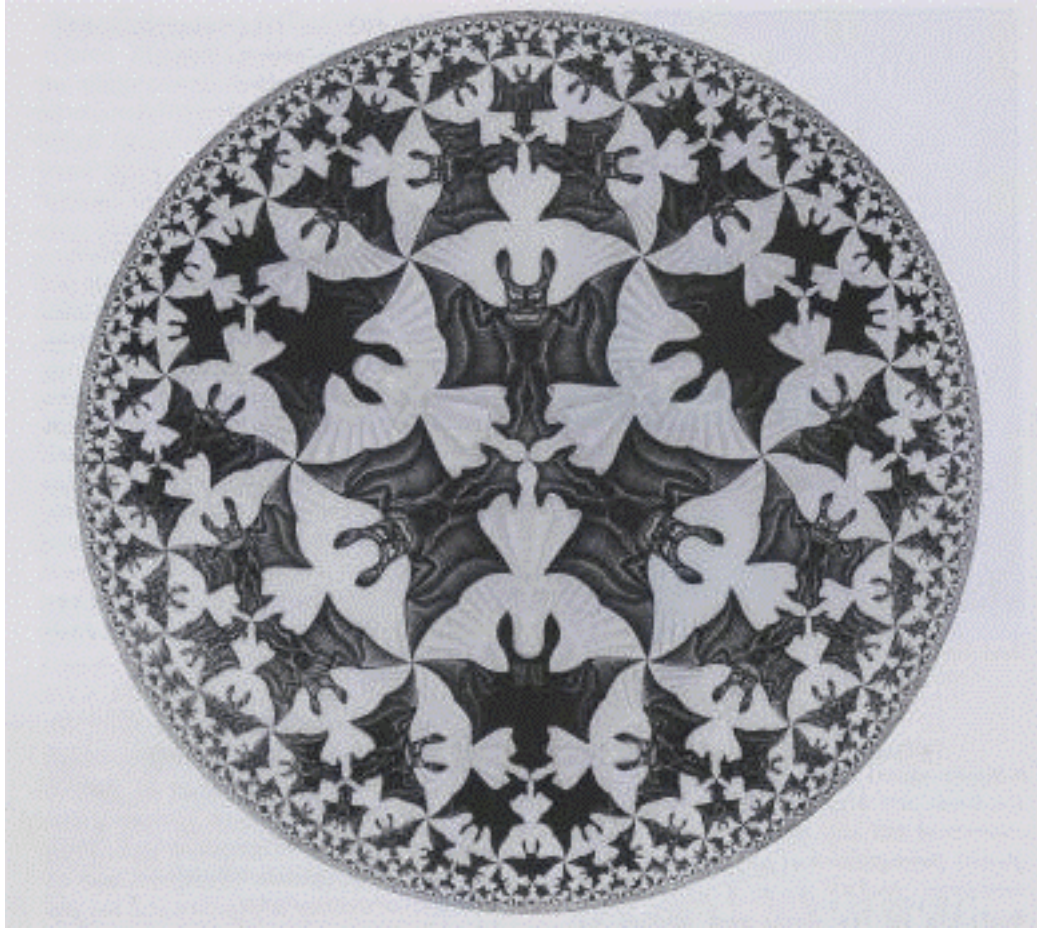
**Proof:** Assume the notation above and that  $PA = mPB$  where  $m > 1$  is a constant. There are two points on  $\overline{AB}$  indicated by  $C$  and  $D$  in the figure with the desired ratio. By the converse to Theorem 2.9.6 and the converse to Theorem 2.9.8,  $\overline{PC}$  and  $\overline{PD}$  are the internal and external angle bisectors of the angle at  $P$ . Thus they are perpendicular (why?), so  $\angle CPD$  is a right angle. This means that  $P$  lies on a circle with diameter  $\overline{CD}$ . **QED**



In the previous proof what happens in the case where  $m < 1$ ? Also, see **Exercise 2.11.2** .

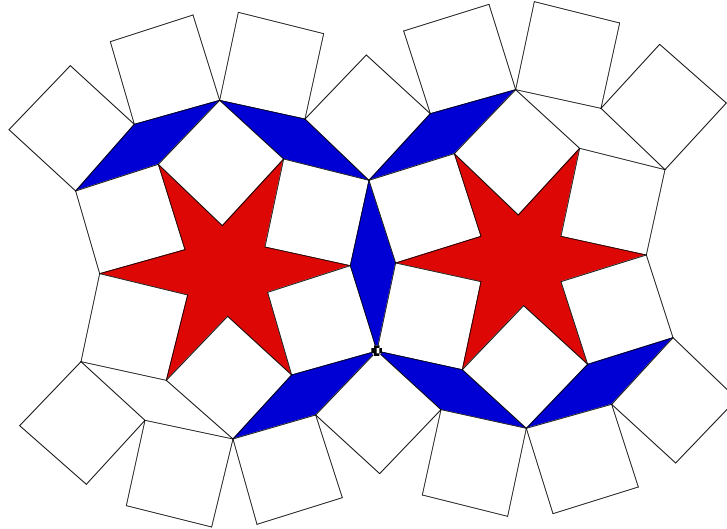
**2.10 TILINGS OF THE EUCLIDEAN PLANE.** The appeal of many of the most interesting decorations or constructions we see around us, whether manufactured or in nature, is due to underlying symmetries. Two good illustrations of this are the so-called ‘Devils and Angels’ designs by the Dutch graphic artist M. C. Escher. Underlying both is the idea of tilings of the plane, in the first example the Euclidean plane, in the second example the hyperbolic plane.





But examples can be found everywhere from floor coverings, to wallpaper, to the mosaics of Roman villas and to decorations of structures as varied as Highway 183 in Austin and Islamic mosques. An understanding of the geometry underlying these designs and their symmetries increases our understanding and appreciation of the artistic design as well as geometry itself. The classification of these symmetries is actually a fascinating problem linking both algebra and geometry, as we shall see later.

Some of the simplest, yet most striking designs come from ‘tilings’ by regular polygons or by congruent polygons. Examples can be found everywhere in Islamic art because of the ban imposed by the Koran on the use of living forms in decoration and art. This style of ornamentation is especially adapted to surface decoration since it is strongly rooted in Euclidean plane geometry. Sketchpad will enable us to reproduce these complicated and colorful designs. Once the underlying geometry has been understood, however, we can make our own designs and so learn a lot of Euclidean plane geometry in the process. Four examples illustrate some of the basic ideas.

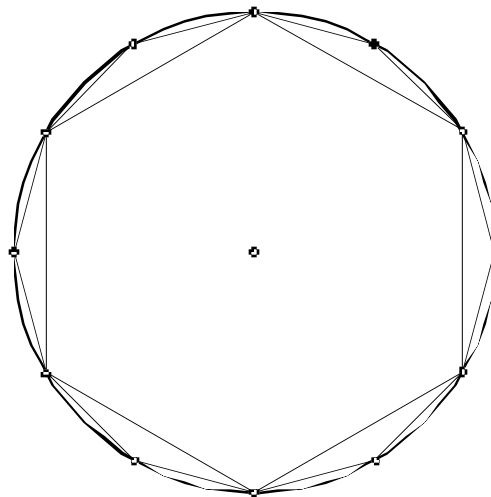


Example 1

The above example shows a typical Arabic design. This was drawn starting from a regular hexagon inscribed in a circle.

**Demonstration 2.10.0.** Construct the design in Example 1 using Sketchpad.

- First draw a regular hexagon and its circumscribing circle. Now construct a regular 12-sided regular polygon having the same circumscribing circle to give a figure like the one below.

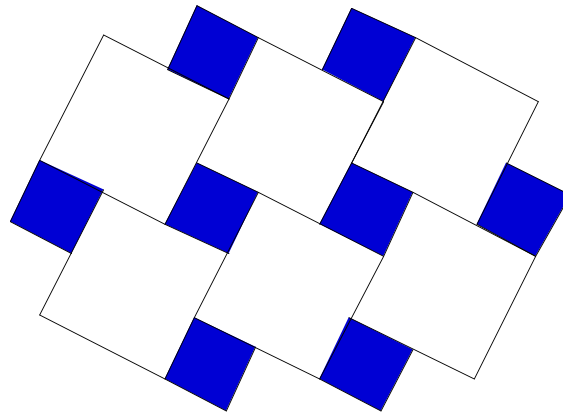


- To construct a second 12-sided regular polygon having one side adjacent to the first regular hexagon, reflect your figure in one of the sides of the first regular hexagon. Now complete the construction of the previous Arabic design.

**End of Demonstration 2.10.0.**

**2.10.1 Exercise.** If the radius of the circumscribing circle of the initial regular hexagon is  $R$ , determine algebraically the area of the six-pointed star inside one of the circles.

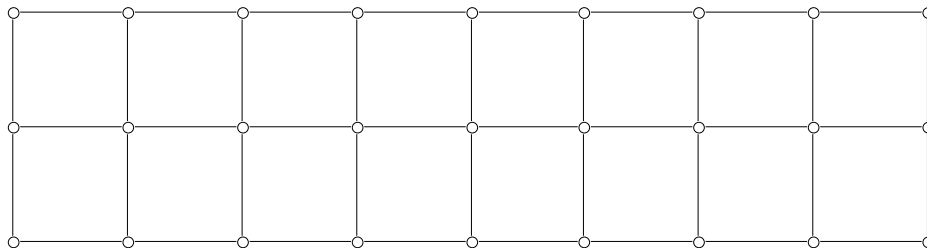
Continuing this example indefinitely will produce a covering of the plane by congruent copies of three polygons - a square, a rhombus and a six-pointed star. Notice that all these congruent copies have the same edge length and adjacent polygons meet only at their edges, *i.e.*, the polygons do not overlap. The second example



Example 2

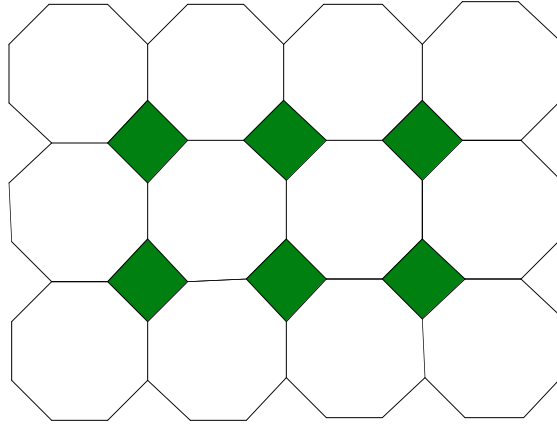
if continued indefinitely also will provide a covering of the plane by congruent copies of two regular polygons - two squares, in fact. Again adjacent polygons do not overlap, but now the individual tiles do not meet along full edges.

The next example



Example 3

is one very familiar one from floor coverings or ceiling tiles; when continued indefinitely it provides a covering of the plane by congruent copies of a single, regular polygon - a square. But now adjacent polygons meet along the full extent of their edges. Finally, notice that continuations of the fourth example



Example 4

produce a covering of the plane by congruent copies of two regular polygons, one a square the other an octagon; again the covering is edge-to-edge.

To describe all these possibilities at once what we want is a general definition of coverings of the plane by polygons without overlaps. Specializations of this definition can then be made when the polygons have special features such as the ones in the first four examples.

**2.10.2. Definition.** A *tiling* or *tessellation* of the Euclidean plane is a collection  $T_1, T_2, \dots, T_n$ , of polygons and their interiors such that

- no two of the tiles have any interior points in common,
- the collection of tiles completely covers the plane.

When all the tiles in a plane tiling are congruent to a single polygon, the tiling is said to have *order one*, and the single region is called the *fundamental region* of the tiling. If each tile is congruent to one of  $n$  different tiles, also called *fundamental regions*, the tiling is said to have *order n*.

Now we can add in special conditions on the polygons. For instance, when the polygons are all regular we say that the tiling is a *regular tiling*. Both the second, third and fourth examples above are regular tilings, but the first is not regular since neither the six-pointed polygon nor the rhombus is regular. To distinguish the second example from the others we shall make a crucial distinction.

**2.10.3. Definition.** A tessellation is said to be *edge-to-edge* if two tiles intersect along a full common edge, only at a common vertex, or not at all.

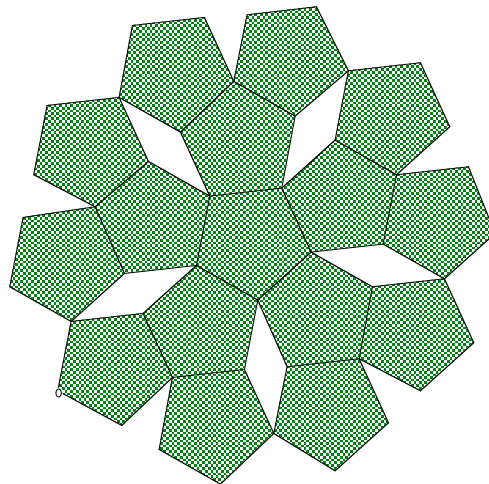
Thus examples one, three and four are edge-to-edge, whereas example two is not edge-to-edge. The point of this edge-to-edge condition is that it reduces the study of regular tilings to

combinatorial problems for the interior angles of the regular polygons meeting at a vertex. It is in this way that the Euclidean plane geometry of this chapter, particularly the sums of angles of polygons, comes into play. So from now on a tiling will always mean an edge-to-edge tiling unless it is explicitly stated otherwise.

A major problem in the theory is to determine whether a given polygon can serve as fundamental region for a tiling of order one, or if a collection of  $n$  polygons can serve as fundamental regions for a tiling of order  $n$ . The case of a square is well-known from floor coverings and was given already in example 3 above.

**2.10.4. Demonstration.** Investigate which regular polygons could be used to create an edge-to-edge regular tiling of order one.

Use the '3/Triangle (By Edge)' script to show that an equilateral triangle can tile the plane meaning that it can serve as fundamental region for a regular tiling of order one. Try the same with a regular hexagon using the '6/Hexagon (By Edge)' script - what in nature does your picture remind you of? Now use the '5/Pentagon (By Edge)' to check if a regular pentagon can be used a fundamental region for a regular tiling of order one. Experiment to see what patterns you can make. One example is given below; can you find others?



**End of Demonstration 2.10.4.**

Can you tile the plane with a regular pentagon? To see why the answer is no we prove the following result.

**2.10.5. Theorem.** The only regular polygons that tile the plane are equilateral triangles, squares and regular hexagons. In particular, a regular pentagon does not tile the plane.

**Proof.** Suppose a regular  $p$ -sided polygon tiles the plane with  $q$  tiles meeting at each vertex.

Since the interior angle of a regular  $p$ -sided polygon has measure  $180\left(\frac{p-2}{p}\right)$ , it follows that

$q180(1 - 2/p) = 360$ . But then

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad \text{i.e., } (p-2)(q-2) = 4.$$

The only integer solutions of this last equation that make geometric sense are the pairs

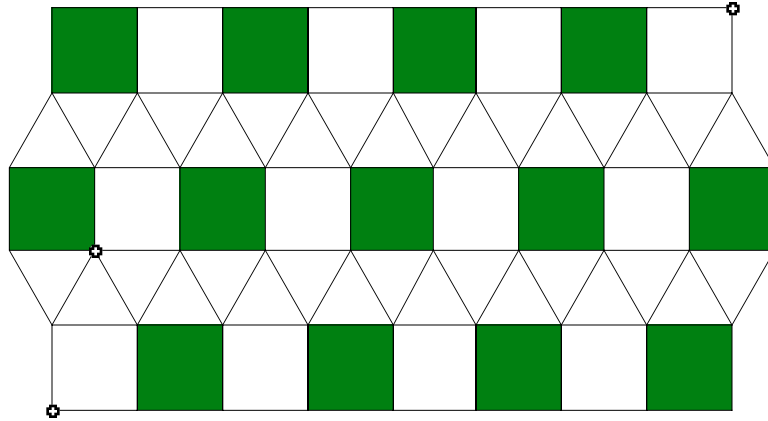
$$(p, q) = (3, 6), (4, 4), \text{ or } (6, 3).$$

These correspond to the case of equilateral triangles meeting 3 at each vertex, squares meeting 4 at each vertex and regular hexagons meeting 3 at each vertex. **QED**

Tilings of the plane by congruent copies of a regular polygon does not make a very attractive design unless some pattern is superimposed on each polygon - that's a design problem we shall return to later. What we shall do first is try to make the tiling more attractive by using more than one regular polygon or by using polygons that need not be regular. Let's look first at the case of an equilateral triangle and a square each having the same edge length.

**Demonstration 2.10.5a.** Construct a regular tiling of order 2 where the order of the polygons is preserved at each vertex.

- Open a new sketch and draw a square (not too big since this is the starting point) and draw an equilateral triangle on one of its sides so that the side lengths of the triangle and the square are congruent. Use the scripts to see if these two regular polygons can serve as the fundamental regions of a regular tiling of order 2 where the order of the polygons is preserved at each vertex. Here's one such example.



Notice that the use of colors can bring out a pattern to the ordering of the polygons at each vertex. As we move in counter-clockwise order around each vertex we go from

$$S(\text{green}) \rightarrow S(\text{yellow}) \rightarrow T(\text{white}) \rightarrow T(\text{blue}) \rightarrow T(\text{white})$$

(and then back to  $S(\text{green})$ ) where  $S$  = square and  $T$  = equilateral triangle. This is one example of an edge-to-edge regular tiling of order two. Consider how many are there.

**End of Demonstration 2.10.5a.**

**2.10.6 Theorem.** Up to similarity there are exactly eight edge-edge regular tilings of order at least 2, where the cyclic order of the polygons is preserved at each vertex.

Keeping the order  $S \rightarrow S \rightarrow T \rightarrow T \rightarrow T$  of squares and triangles produced one such tiling. Convince yourself that  $S \rightarrow T \rightarrow T \rightarrow S \rightarrow T$  produces a different tiling. Why are these the only two possible orderings for two squares and three triangles? How many permutations are possible for the letters  $S, S, T, T,$  and  $T$ ?

What are the other six tilings? Algebraic conditions limit drastically the possible patterns so long as the tiling is edge-to-edge and that the order of the polygons is the same at each vertex. Using the angle sum formulas for regular polygons one can easily see that you need at least three polygons around a vertex, but can have no more than six. In the case of a  $p$ -gon, a  $q$ -gon, and an  $r$ -gon at each vertex, you get the equation

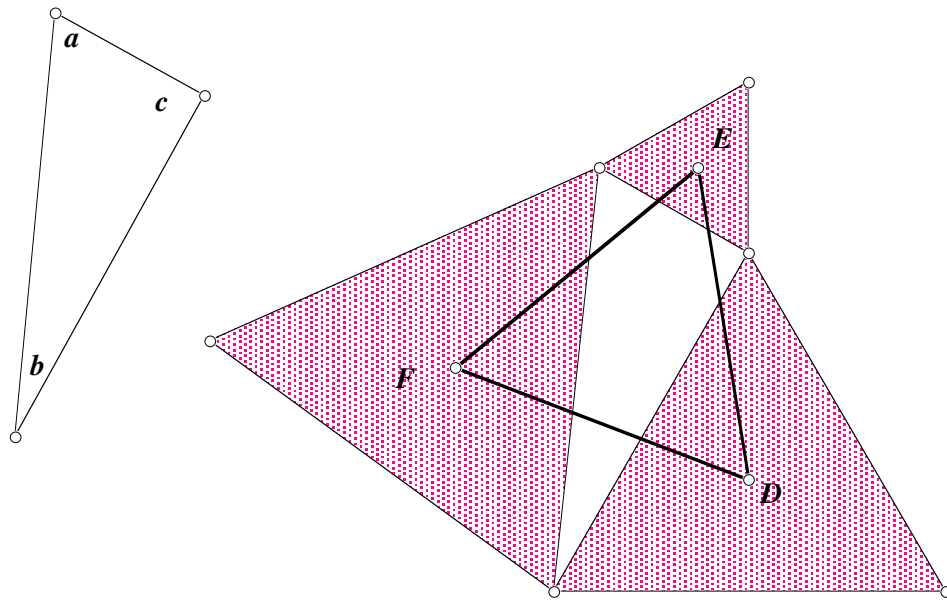
$$180 \left( \frac{p-2}{p} \right) + 180 \left( \frac{q-2}{q} \right) + 180 \left( \frac{r-2}{r} \right) = 360$$

You can check that  $(4,8,8)$ ,  $(4,6,12)$ , and  $(3,12,12)$  are solutions. (There are a few other solutions as well, but they will not make geometric sense.) Thus  $S \rightarrow O \rightarrow O$ ,  $S \rightarrow H \rightarrow D$ , and  $T \rightarrow D \rightarrow D$  all produce tilings, where  $O$  stands for Octagon,  $H$  for hexagon, and  $D$  for



Dodecagon. We are still missing three tilings, but you can have fun looking for them! (See **Exercise 2.11.3.**) Now we will take a look at some less regular tilings.

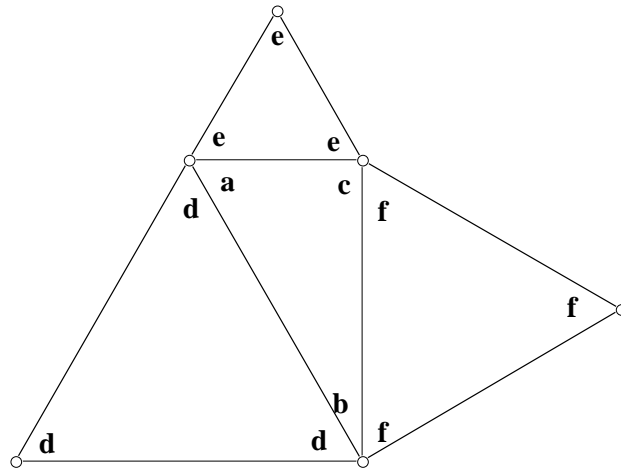
It is surprising how much of geometry can be related to tilings of the plane. Let's consider two instances of this, the second being Pythagoras' theorem. The first instance is a theorem known familiarly as Napoleon's theorem after the famous French general though there is no evidence that he actually had anything to do with the theorem bearing his name! Recall that earlier we proved the form of Pythagoras' theorem saying that the area of the equilateral triangle on the hypotenuse is equal to the sum of the areas of the equilateral triangles on the other two sides. On the other hand, Napoleon's theorem says that the centers of these three equilateral triangles themselves form an equilateral triangle, as we saw in Exercise 1.8.5. The figure below makes this result clearer.



Here  $D$ ,  $E$ , and  $F$  are the centers of the three equilateral triangles where by *center* is meant the common circumcenter, centroid and orthocenter of an equilateral triangle. Napoleon's theorem says that  $\triangle DEF$  is equilateral - it certainly looks as if its sides are congruent and measuring them on Sketchpad will establish congruence. You will provide a proof of the result in Exercise 2.11.5. The question we consider here is how all this relates to tilings of the plane. Notice now that we have labeled the interior angles of the triangle because we are going to allow polygons which are not necessarily regular. Since the interior can then be different, the particular interior angle of polygons that appears at a vertex is going to be just as important as

which polygon appears. Now we will see how we can continue the figure above indefinitely and thus tile the plane.

One should notice that the edge-to-edge condition imposes severe restrictions on the angles that can occur at a vertex. Label the angles in the original figure as follows.



Of course, the angles of the equilateral triangles are all the same but we have used different letters to indicate that they are the interior angles of equilateral triangles of different size. Since  $a + b + c + d + e + f = 360^\circ$ , three copies of the right-angled triangle and one copy of each of the three different sizes of equilateral triangle will fit around a vertex with no gaps or overlaps. The figure can thus be constructed indefinitely by maintaining the same counter-clockwise order  $a \rightarrow e \rightarrow c \rightarrow f \rightarrow b \rightarrow d$  at each vertex. Now draw the figure for yourself! It may be instructive to use a different color for each equilateral triangle to highlight the fact that the equilateral triangles are not necessarily congruent.

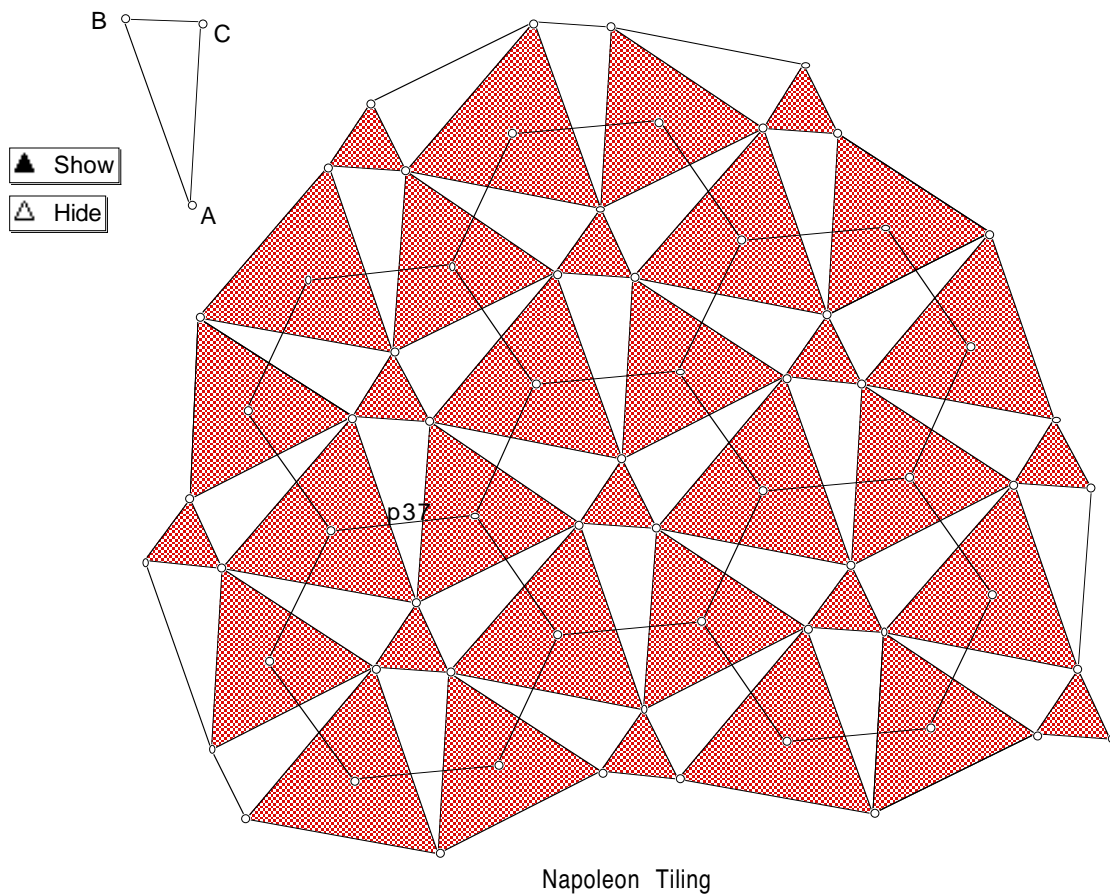
### 2.10.6a Demonstration.

- Open a new sketch and in the top left-hand corner of the screen draw a right-angled triangle as shown in the figure above. Make sure that your construction is dynamic in the sense that the triangle remains right-angled whenever any one of the vertices is dragged.
- Use the ‘Circle By Center + Radius’ construction to construct a congruent copy of your triangle in the center of the screen. Draw an outwardly pointing equilateral triangle on each side of this right-angled triangle.

- Continue adding congruent copies of the right-angled triangle and the equilateral triangles to the sides of the triangles already in your figure. (One way to add congruent copies of the right triangle is to use your ‘Auto-Matching’ similar triangle script. Just label your original right triangle appropriately.)
- Experiment a little to see what figures can be produced. Check that your construction is dynamic by dragging the vertices of the first right-angled triangle you drew.

**End of Demonstration 2.10.6a.**

Here’s one that looks as if it might tile the plane if continued indefinitely.

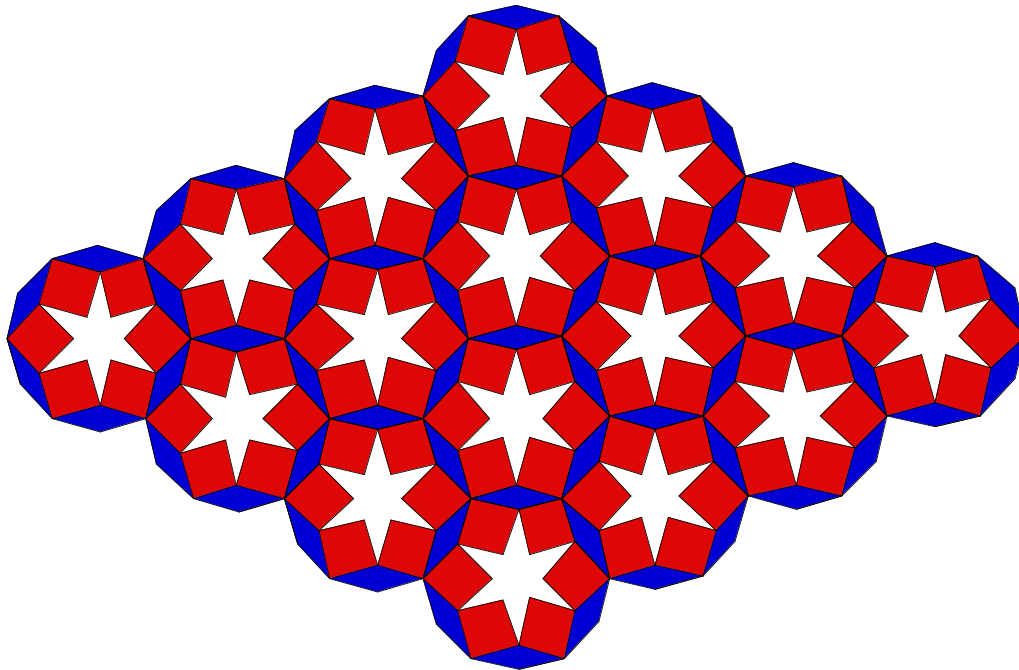


Napoleon Tiling

The figure above of the Napoleon Tiling has an overlay of hexagons over it. To see where it came from, apply Napoleon’s Theorem to the tiling. That is around each right triangle connect the centers of the equilateral triangle to create a new equilateral triangle. Six of those new equilateral triangles make up each hexagon above. Thus Napoleon’s theorem brings out an *underlying* symmetry in the design because it showed that a regular tiling of the plane by

regular hexagons could be overlaid on the figure. The same design could have been obtained by putting a *design* on each regular hexagon and then tiling the plane with these patterned regular hexagons.

This brings out a crucial connection between *tilings* and the sort of *designs* that are used for covering walls, floors, ceilings or any flat surface. A design is said to be *wallpaper* design if a polygonal portion of it provides a tiling of the plane by translations in two different directions. Thus all the examples obtained in this section are wallpaper designs. It is very clear that the Islamic design in problem 2.10.1 is a wall-paper design because the portion of the design inside the initial regular hexagon will tile the plane as the figure below clearly shows.



**2.10.7. Exercise.** Find a square portion of Example 4 in Section 2.10 that tiles the plane. In other words, show that that example is a wallpaper design.

Example 2 is sometimes called the “Pythagorean Tiling”. It is created by a translation of two adjacent non-congruent squares. This tiling occurs often in architectural and decorative designs as seen in this sidewalk tiling. To see why this tiling might be called a “Pythagorean Tiling” open a new sketch and draw the tiling as it appears in example 2 using two squares of different sizes. Construct an overlaying of this design by a tiling, which consists of congruent copies of a single square. What is the area of this square? Use Pythagoras’ theorem to relate this area to the area of the two original squares you used to construct your pattern.

## 2.11 Exercises.

**Exercise 2.11.1.** Prove Theorem 2.9.4. Given a circle  $\Sigma$  and a point  $P$  outside  $\Sigma$ , let  $l$  be a ray through  $P$  intersecting  $\Sigma$  at points  $A$  and  $B$ . If  $C$  is a point on  $\Sigma$  such that  $PA \cdot PB = PC^2$ , then  $\overline{PC}$  is tangent to  $\Sigma$  at  $C$ .

**Exercise 2.11.2:** Given points  $A$ ,  $B$  and  $P$  use Sketchpad to construct the Circle of Apollonius passing through  $P$ . In other words, construct the set of points  $Q$  such that  $QA = mQB$  where  $PA/PB = m$ .

**Exercise 2.11.3.** Produce two different order-preserving edge-to-edge regular tilings of order 2, just using triangles and hexagons. Produce an order-preserving edge-to-edge regular tiling of order 3 using triangles, squares, and hexagons. (We now have the eight tilings mentioned in Theorem 2.10.5!)

**Exercise 2.11.4.** Using Sketchpad construct the Napoleon Tiling. Choose a regular hexagon in your figure and describe its area in terms of the original triangle and the three equilateral triangles constructed on its sides. Now choose a different (larger or smaller area) regular hexagon having a different area and describe the area of this hexagon in terms of the original triangle and the three equilateral triangles.

**Exercise 2.11.5.** While the tiling above makes a very convincing case for the truth of Napoleon's theorem it doesn't prove it in the usual meaning of 'proof'. Here is a coordinate geometry proof based on the figure on the following page and on the notation in that figure.

(a) The points  $D$ ,  $E$ , and  $F$  are the centers of the equilateral triangles constructed on the sides of the right-angled triangle  $\triangle ABC$ . Show that length  $\overline{BF} = c \sqrt{3}$ . Determine also the lengths of  $\overline{AD}$  and  $\overline{BE}$ .

(b) If  $\angle ABC = \theta$  and  $\angle CAB = \phi$ , write the values of  $\sin \theta$ ,  $\cos \theta$ ,  $\sin \phi$ , and  $\cos \phi$  in terms of  $a$ ,  $b$ , and  $c$ .

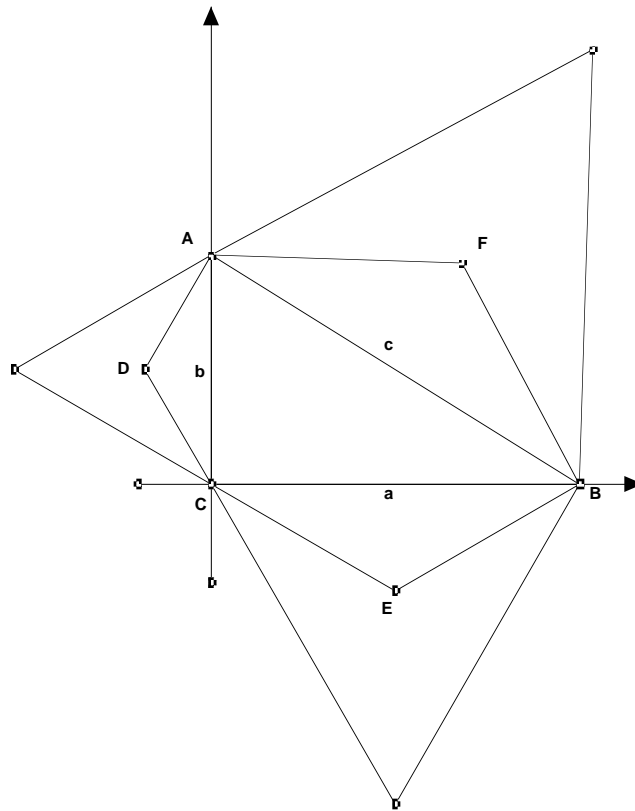
(c) Write down the addition formulas for sine and cosine.

$$\cos(u + v) = \quad , \quad \sin(u + v) = \quad .$$

(d) Let the lengths of  $\overline{FE}$ ,  $\overline{DF}$ , and  $\overline{DE}$  be  $x$ ,  $y$  and  $z$  respectively. Use the Law of Cosines to show that

$$z^2 = \frac{1}{3}(a^2 + b^2 + 2ab \cos 30^\circ).$$

Determine corresponding values for  $x$  and  $y$ . Deduce that  $x = y = z$ .

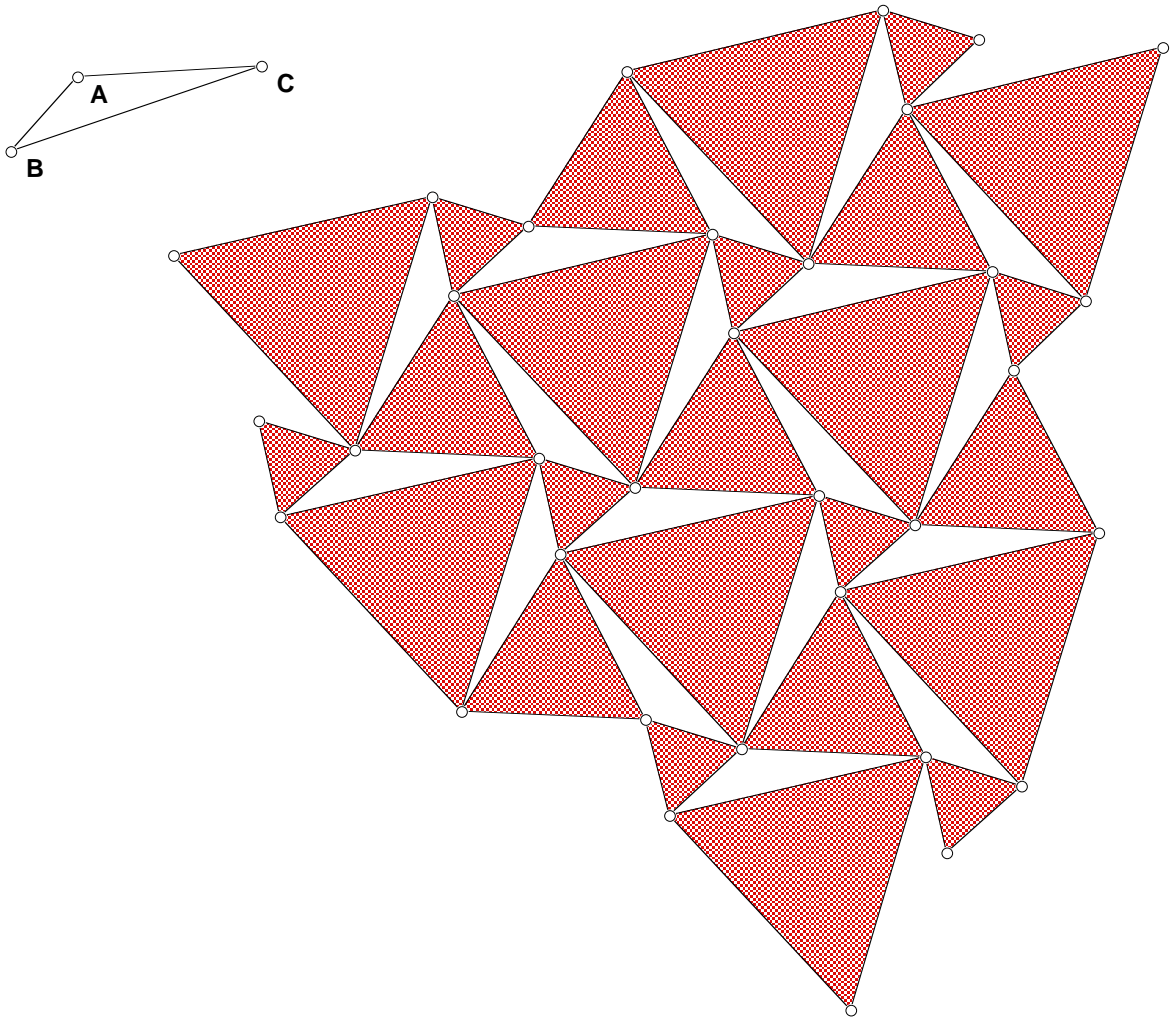


Use all the previous results to finish off a coordinate geometry proof of Napoleon's theorem.

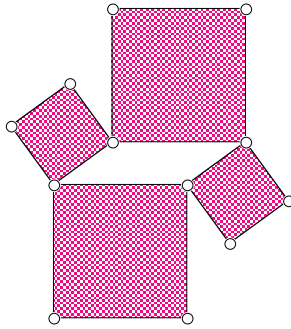
**Exercise 2.11.6.** Instead of starting with a right-angled triangle, start with an arbitrary  $\triangle ABC$  and draw equilateral triangles on each of its sides and repeat the previous construction.

- Open a new sketch and draw a small triangle near the top corner of the screen; label the vertices  $A$ ,  $B$ , and  $C$ . By using the 'Circle By Center+Radius' tool you can construct congruent copies of this triangle.

- Draw one congruent copy of  $\triangle ABC$  in the center of the screen. Draw an equilateral triangle on each of its sides.
- Continue this construction preserving cyclic order at each vertex to obtain a tiling of the plane. The following figure is one such example.
- Construct the centers of all the equilateral triangles and draw hexagons as in the case of right-angled triangles. Do you think Napoleon's theorem remains valid for any triangle, not just right-angled triangles?

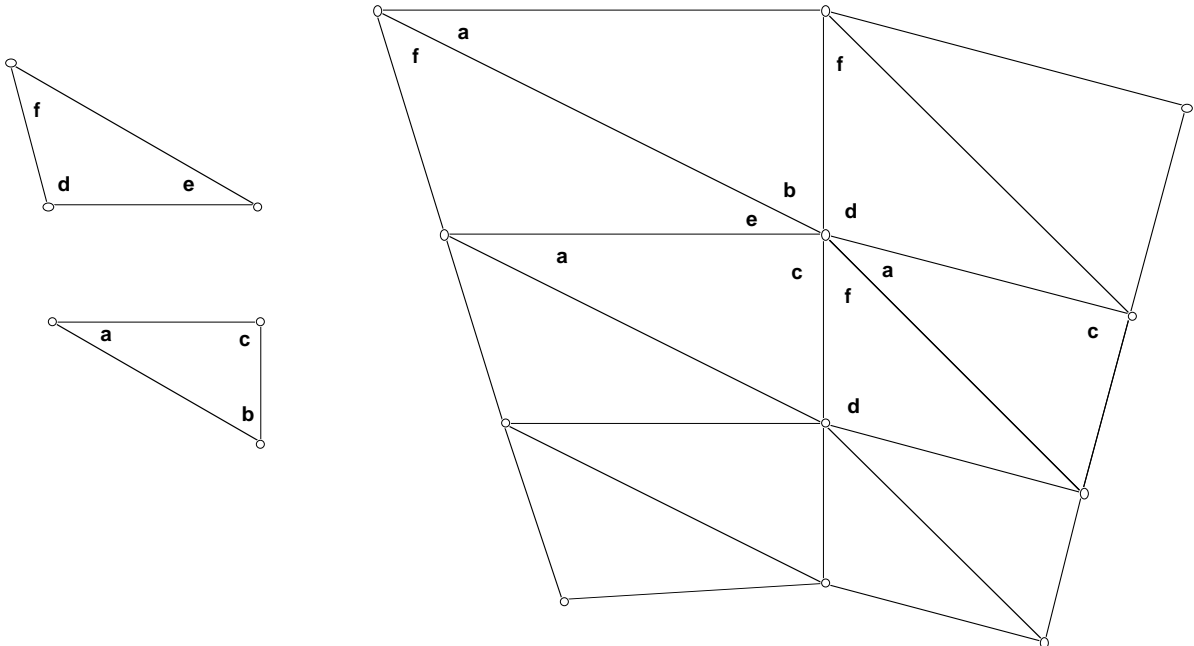


**Exercise 2.11.7.** Can the plane be tiled by copies of the diagram for Yaglom's Theorem (given below) as in the manner of the tiling corresponding to Napoleon's Theorem? If so, produce the tiling using Sketchpad. Recall that Yaglom's Theorem said if we place squares on the sides of a parallelogram, the centers of the squares also form a square.



## 2.12 One Final Exercise.

**Exercise 2.12.1.** To the left in the figure below are two triangles, one obtuse, the other right-angled. The interior angles of the two triangles have been labeled. Since the sum of these six angles is  $360^\circ$  there should be a tiling of the plane by congruent copies of these two triangles in which the cyclic order of the angles at each vertex is the same as the one shown in the figure to the right.



- Open a new sketch and continue this construction to provide a tiling of the plane. Unlike the previous tilings, the triangles in this tiling are not congruent. Explain why this tiling is more like a Nautilus Shell.



- Construct the circumcenters of the three outwardly pointing obtuse triangles on the sides of one of the right-angled triangles and join these circumcenters by line segments. What, if any, is the relation of the triangle having these three circumcenters as vertices and the original obtuse triangle? Is there any relation with the original right-angled triangle? Use Sketchpad if necessary to check any conjecture you make. (Don't forget to drag!)

Investigate what happens if you construct instead the three circumcenters of the right-angled triangles on the sides of one of the obtuse triangles? Draw the triangle having these circumcenters as vertices. What, if any, is the relation between the original right-angled triangle and the triangle having the three circumcenters as vertices? Is there any relation with the original obtuse triangle? Again use Sketchpad if necessary to check visually any conjecture you make. (Don't forget to drag!)

