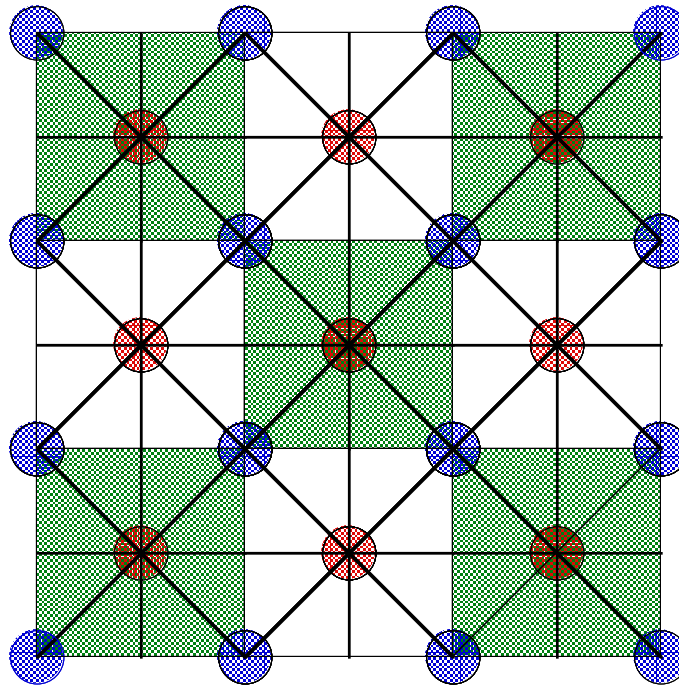


Let's go back to our checkerboard design - we shall think of it as extending over the whole plane to form a tiling by congruent copies of a single square. An alternating coloring has been added for extra effect. This tiling will be left unchanged by various reflections and rotations about various points.

- Go back to the checkerboard figure and mark in the mirror lines with respect to which a reflection leaves the design unchanged. Mark the mirror lines in bold. Mark in red the centers of rotation through  $90^\circ$  that leave the design unchanged. Mark also in blue the centers of rotation through  $180^\circ$  that leave the design unchanged.
- Your pattern should look like the one below.



- This tiling is classified as “p4m”. The smallest rotations allowed are quarter -turns and there are reflections in four directions.
- Successive use of the reflections and rotations fixing the design would replicate the whole tiling from just one white square and one colored tile. Can the whole tiling be generated from any part smaller than these two squares? Find the smallest piece from which the whole tiling could be generated by successive reflections and rotations. This smallest piece is called a *Fundamental Domain*.

How would the pattern of reflections and rotations differ if the tiling consisted of all white squares? What is a Fundamental domain of the new monochromatic tiling?

You can continue to examine wallpaper designs in the next set of exercises. Now we will assemble all the results and ideas developed about transformations and tilings to show how to use Sketchpad to construct figures with a prescribed symmetry. First let's see how to use Custom Tools to define our own transformations.

#### 4.4.1 Demonstration. Custom transformations.

A custom transformation is a sequence of one or more transformations. The basic steps are given below.

- Transform an object one or more times.
- Hide any intermediate objects or format them as you wish them to appear when you apply your transformation.
- Select the pre-image and image, and select and show the labels of all marked transformation parameters.
- Create a new tool. The pre-image and transformation parameters will become given objects in the custom tool.
- In the custom tool's Script View, set each of the given transformation parameters—mirrors, centers, and so forth—to automatically match objects with the same label..

For example, let's define a rotation  $\rho_{A,\theta}$  through a given angle  $\theta$  about a given point  $A$ .

- Open a new sketch and construct a point  $A$  and any point  $P$ . Mark  $A$  as a center of rotation. Then construct the point  $P'$  which is the rotation of  $P$  about  $A$  through an angle  $\theta$  (choose any  $\theta$ ).
- Next select  $P$  and  $P'$  and  $A$ . Choose "Create New Tool from the **Tools** menu. Type a name that describes the transformational sequence. In the Script View window, double click on Point  $A$  in the "Given" section and check the box "Automatically Match Sketch Object".

- You can now apply your custom transformation to any figure in your sketch. Draw any polygonal figure in your sketch and construct its interior. Select the polygon interior and apply the tool.

Repeat this process to define a reflection  $S_m$  about a given mirror line  $m$  and a translation  $T_v$  in a given direction.

#### **End of Demonstration 4.4.1.**

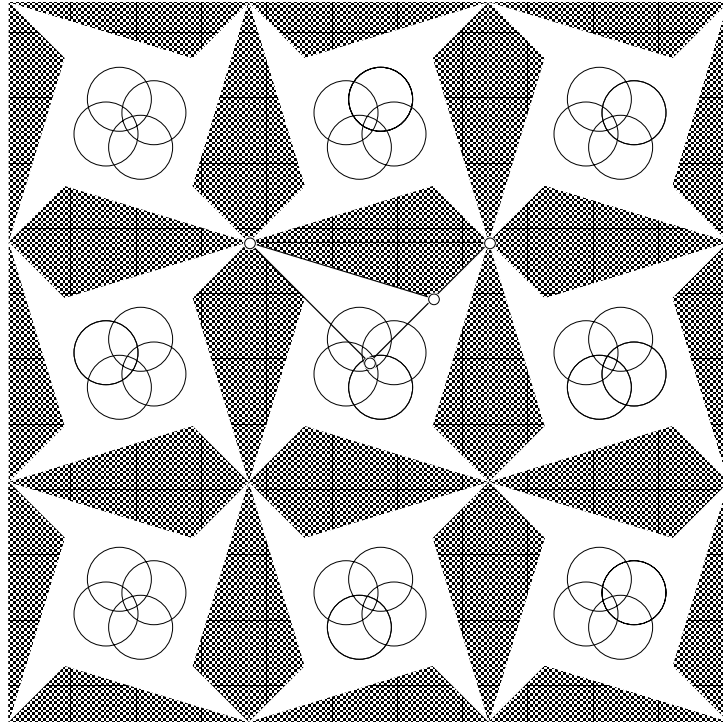
When you define a multi-step transformation, Sketchpad remembers the formatting you've applied to each step's image—whether you've colored it, or hidden it, and so forth. When you apply the transformation to new objects, Sketchpad creates intermediate images with exactly the same formatting. If you are interested only in the final image of the sequence of transformational steps, and not in the intermediate images, hide each intermediate image between your two selected objects before defining the transformation. If you want your transformed images to have a certain color, then be sure your image has the appropriate color when you define the transformation.

#### **4.4.2 Demonstration. Producing a picture with $p4g$ symmetry.**

To utilize these ideas and generate the symmetries necessary for producing a picture having  $p4g$  symmetry:

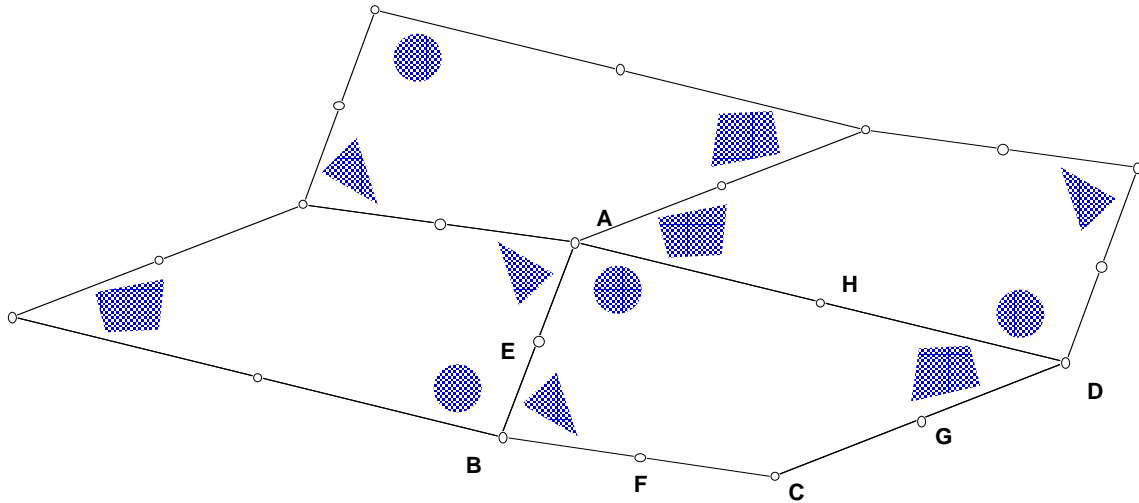
- Create a tool which performs a 4-fold rotation about  $A$ ; call it *4-foldrot*. Construct a 2-fold rotation about  $B$ ; call it *2-foldrot*. Finally construct a reflection about the side  $\overline{BC}$  of  $\triangle ABC$ .
- Construct a right-angled isosceles triangle  $\triangle ABC$  having a right angle at  $A$ ; this will be the fundamental domain of the figure.

Now you are free to draw any figure having  $p4g$  symmetry. Below is one example. The original  $\triangle$  has been left in. The picture was constructed from one triangle inside the fundamental domain and one circle. The most interesting designs usually occur when the initial figure 'pokes' outside the fundamental domain. The vertices of the original triangle can be dragged to change the appearance of the design; the original design can be dragged too. This often results in a radical change in the design.



### End of Demonstration 4.4.2

Earlier, as a consequence of the Euclidean parallel postulate, we saw that the sum of the angles of a triangle is always  $180^\circ$  no matter the shape of the triangle; similarly the sum of the angles of a quadrilateral is always  $360^\circ$  no matter the shape of the quadrilateral. Somewhat later we gave a more careful proof of this fact by determining the sum of the angles of any polygon - in fact we saw that the value depends only on the number sides of the polygon. This value was then used to show that equilateral triangles, squares and regular hexagons are the only regular polygons that tile the Euclidean plane. But nothing was said about the possibility of non-regular polygons tiling the plane. In fact, any triangle or quadrilateral can tile the plane. The figure below illustrates the case of a convex quadrilateral.  $ABCD$  was the original quadrilateral and  $E, F, G, H$  are the respective midpoints. One can obtain the figure below by rotating by  $180^\circ$  about the midpoint of each side of the quadrilateral. (You can tile the plane with any triangle by the same method – try it!)

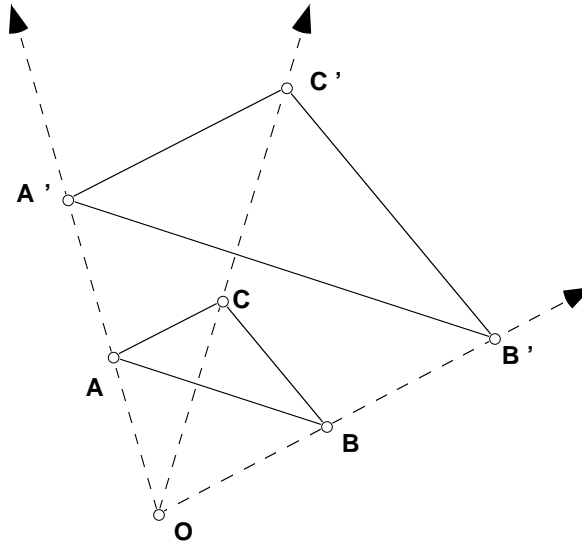


To do this for yourself, you can use custom transformations. Define a transformation for each midpoint. I've drawn a different figure in each of the corners of the chosen quadrilateral to help me distinguish among the corners. Use your four rotations to produce a tiling of the plane by congruent copies of the original quadrilateral with one copy of each of the four corners occurring at every vertex. Join neighboring images of the midpoints by line segments. What resulting repeating diagram emerges? You should see an overlay of parallelograms. Can you find a parallelogram and points so that successive rotations of the parallelogram through  $180^\circ$  about the points would produce the same tiling?

**4.5 DILATIONS.** In this section we would like look at another type of mapping, dilation, that is frequently used in geometry. Dilation will not be an isometry but it will have another useful property, namely that it preserves angle measure.

**4.5.1 Definition.** A geometric transformation of the Euclidean Plane is said to be conformal when it preserves angle measure. That is, if  $A'$ ,  $B'$ , and  $C'$  are the images of  $A$ ,  $B$ , and  $C$  then  $m\angle A'B'C' = m\angle ABC$ .

**4.5.2 Definition.** A dilation with center  $O$  and dilation constant  $k \neq 0$  is a transformation that leaves  $O$  fixed and maps any other point  $P$  to the point  $P'$  on the ray  $OP$  such that  $OP' = k \cdot OP$ .



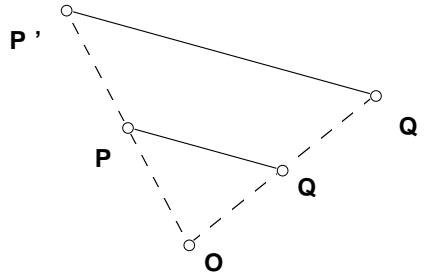
#### 4.5.2a Demonstration. Dilation with Sketchpad.

Sketchpad has the dilation transformation built into the program.

- Open a new sketch and construct a point  $O$  and  $\triangle ABC$ .
- Select  $O$  and then “Mark Center  $O$ .” under the **Transform** menu.
- Select  $\triangle ABC$  and then select “dilate” from the **Transform** menu.
- Enter the desired scale factor (dilation constant). (In the figure above the dilation constant is equal to 2. Notice that in the dialogue box, the scale factor is given as a fraction. In this case, we would either enter  $\frac{2}{1}$  or  $\frac{1}{0.5}$ .)
- What is the image of a segment under dilation? Is the dilation transformation is conformal?
- Next construct a circle and dilate about the center  $O$  by the same constant. What is the image of a circle?

**End of Demonstration 4.5.2a.**

**4.5.3 Theorem.** The image of  $\overline{PQ}$  under dilation is a parallel segment,  $\overline{P'Q'}$  such that  $P'Q' = |k| \cdot PQ$



**Proof.** From SAS similarity it follows that  $\triangle POQ \sim \triangle P'OQ'$  and thus  $P'Q' = |k| \cdot PQ$ . The proof needs to be modified when  $O, P$ , and  $Q$  are collinear.

**4.5.4 Theorem.** The dilation transformation is conformal.

**Proof.** See **Exercise Set 4.6**.

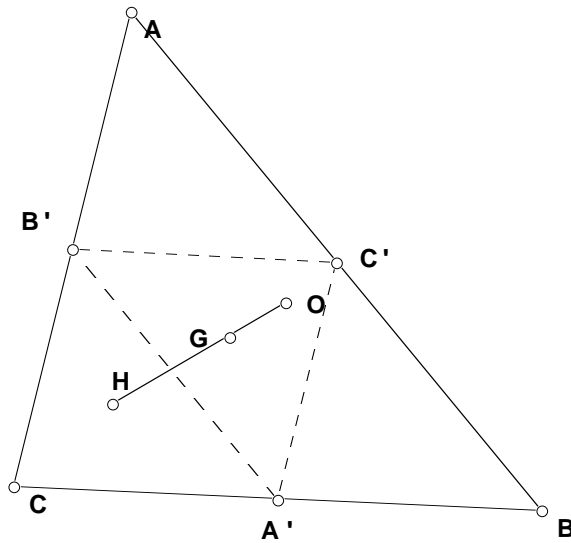
One can easily see that the following theorem is also true. The idea for the proof is to show that all points are a fixed distance from the center.

**4.5.5 Theorem.** The image of a circle under dilation is another circle.

**Proof.** Let  $O$  be the center of dilation,  $Q$  be the center of the circle, and  $P$  be a point on the circle.  $Q'$  will be the center of the image circle. By Theorem 4.5.3,  $\frac{P'Q'}{PQ} = \frac{OQ'}{OQ}$  or

$P'Q' = \frac{PQ \cdot OQ'}{OQ}$ . Now each segment in the right-hand expression has a fixed length so  $P'Q'$  is a constant. Thus for any position of  $P$ ,  $P'$  lies on a circle with center  $Q'$ .

Using dilations we can provide an alternate proof for the fact that the centroid of a triangle trisects the segment joining the circumcenter and the orthocenter (The Euler Line).

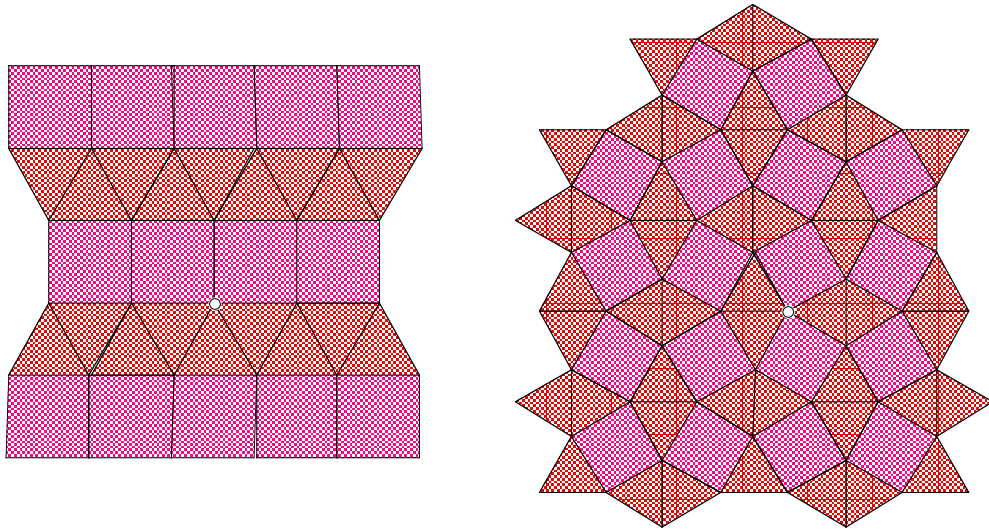


Given  $\triangle ABC$  with centroid  $G$ , orthocenter  $H$ , and circumcenter  $O$ . Let  $A'$ ,  $B'$ , and  $C'$  be the midpoints of the sides. First note that  $O$  is the orthocenter of  $\triangle A'B'C'$  and that  $G$  divides each median into a 2:3 ratio. Thus if we dilate  $\triangle ABC$  about  $G$  with a dilation constant of  $-\frac{1}{2}$ ,  $\triangle ABC$  will get mapped to  $\triangle A'B'C'$  and  $H$  will get mapped to  $O$  (their orthocenters must correspond). Hence  $O$ ,  $G$ , and  $H$  must be collinear by the definition of a dilation and  $OG = \frac{1}{2}HG$ . **QED.**

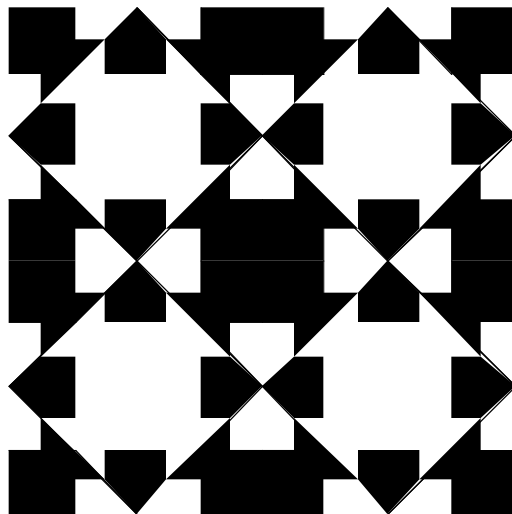
#### 4.6 Exercises.

**Exercise 4.6.1.** Recall the two regular tilings of order 2 produced with squares and triangles. Classify each as a wallpaper design.

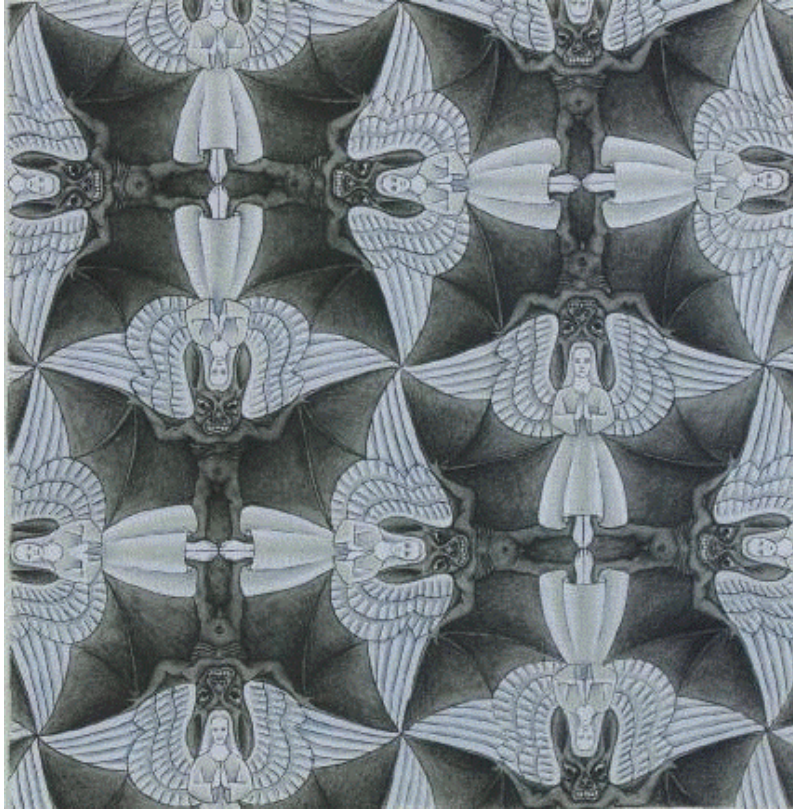




**Exercise 4.6.2.** Classify the following wallpaper design. Is there any relation to the checkerboard tiling?

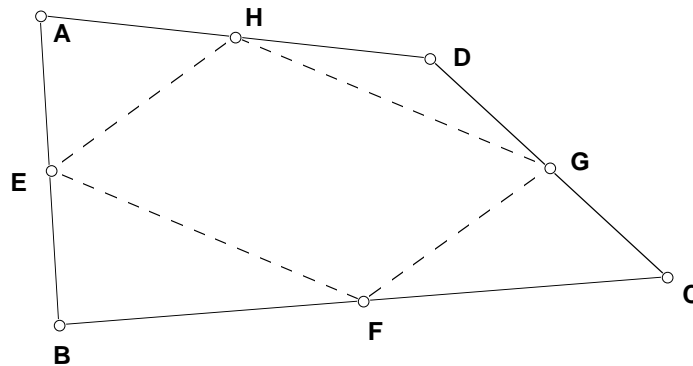


**Exercise 4.6.3.** What type of wallpaper design is Escher's version of 'Devils and Angels' for Euclidean geometry?



**Exercise 4.6.4.** On sketchpad use custom transformations to create a wallpaper design other than a  $p4g$ .

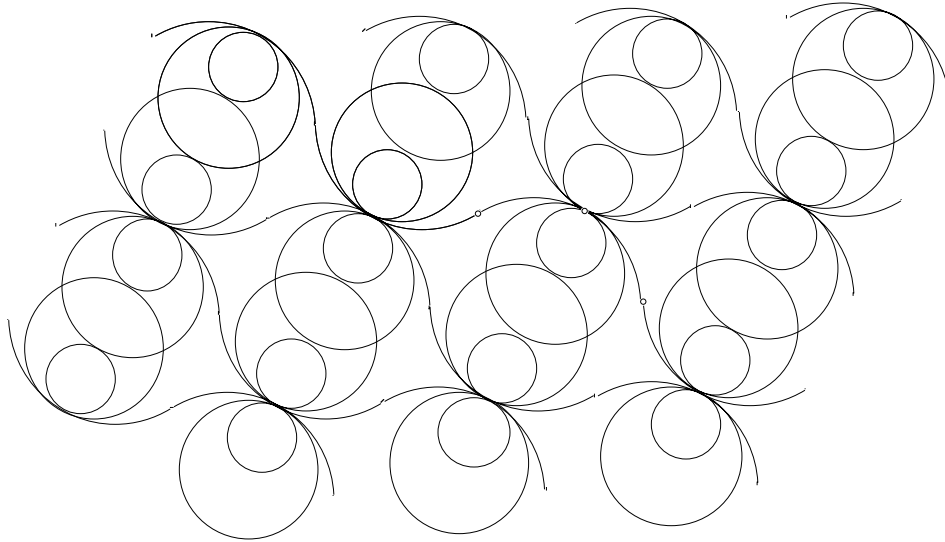
**Exercise 4.6.5.** Let  $ABCD$  be a quadrilateral. In the figure below  $E, F, G,$  and  $H$  are the midpoints of the sides. Prove that  $EFGH$  is a parallelogram. Hint: Similar triangles.



**Exercise 4.6.6.** Escher's lizard graphic is shown below. Mark all the points in the picture about which there are rotations by  $180^\circ$ . What do you notice about these points? Exhibit a parallelogram and three points about which successive rotations through  $180^\circ$  would produce Escher's design. What is the wallpaper classification for the lizard design?



**Exercise 4.6.7.** Now pretend that you are Escher. Start with a parallelogram  $PQRS$ . Draw some geometric design inside this parallelogram - a combination of circles and polygons, say. Choose three points and define rotations through  $180^\circ$  about these points so that successive rotations about these three points tiles the plane with congruent copies of your design. Try making a second design allowing some of the circles and polygons to fall outside the initial parallelogram - this usually produces a more interesting picture. Here's one based on two circles and an arc of a circle

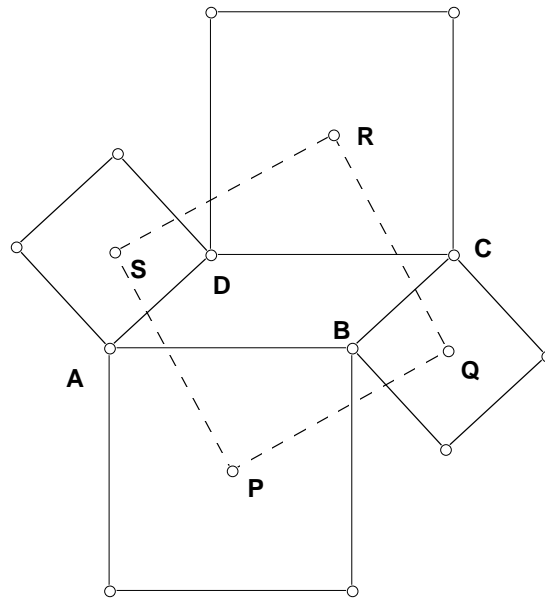


**Exercise 4.6.8.** Prove Theorem 4.5.4. The dilation transformation is conformal.

## 4.7 USING TRANSFORMATIONS IN PROOFS

Transformations can also be useful in proving certain theorems, sometimes providing a more illuminating proof than those accomplished by synthetic or analytic methods. We “discovered” Yaglom’s Theorem in the second assignment and re-visited it while looking at tilings. There is an easy proof that uses transformations.

**4.7.1 Theorem.** Let  $ABCD$  be any parallelogram and suppose we construct squares externally on each side of the parallelogram. Then centers of these squares also form a square.



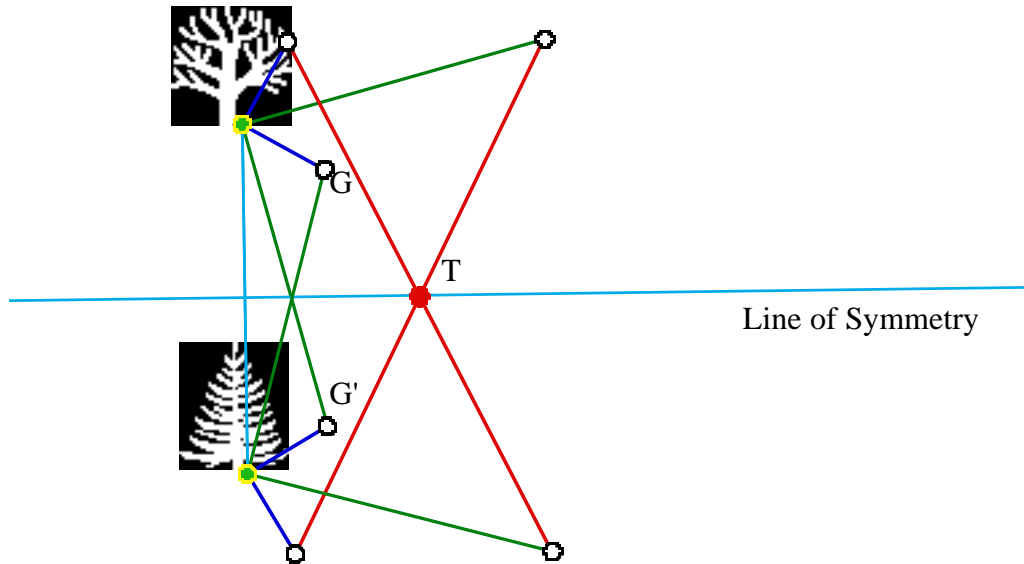
**Proof.** Consider the rotation about  $P$  by  $90^\circ$ . (Try it on sketchpad.) The square centered at  $P$  will rotate onto its original position and  $\overline{AB}$  must rotate to  $\overline{AC}$ , so the square centered at  $Q$  will rotate to onto the square centered at  $S$ . Thus their centers will coincide. This tells us that the segment  $\overline{PQ}$  rotates  $90^\circ$  onto the segment  $\overline{PS}$ , and therefore  $PQ=PS$  and  $m\angle QPS=90$ . Do the same for the other centers  $Q, R,$  and  $S$ . Thus  $PQRS$  is a square.

**QED**

Earlier in this chapter we looked at the Buried Treasure problem (**Exercise 4.3.6**). After working with the Treasure sketch one notices that the location of the treasure is likely to be independent of the position of the gallows. If we use this observation as an assumption, then perhaps we can gain an understanding as to where the treasure is buried with respect to the trees.

The map's instructions are very symmetrical. Since the only reference points are the two trees, a symmetry argument will be used with objects reflected across the perpendicular bisector of the segment joining the trees. Choose a position for the gallows ( $G$ ) near the Oak tree, and its reflection ( $G'$ ) near the Pine tree (Figure 1).

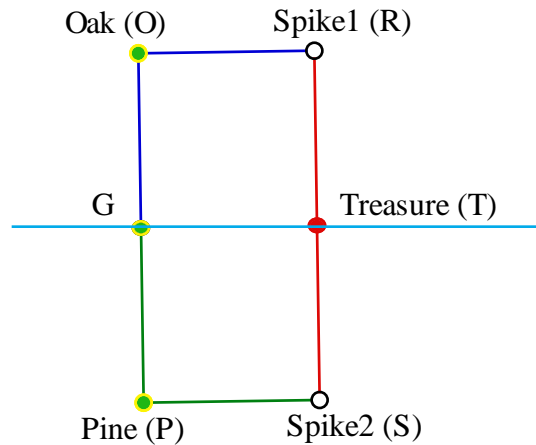
Figure 1



The treasure must lie upon the line of symmetry; or else it is in two different places. Therefore, the treasure lies upon the perpendicular bisector of the Pine Oak segment.

To calculate where upon the perpendicular bisector the treasure lies, we next choose  $G$  to be a point on the line of symmetry, specifically the midpoint between the Pine ( $P$ ) and the Oak ( $O$ ) trees (Figure 2). We will need to find  $GT$ . Since  $G$  is the midpoint of  $\overline{OP}$ , we see that  $GO = GP$ ; in addition, by following the treasure map directions, we see that  $GP = PS$  and  $GO = OR$ .

Figure 2



$OR = PS$  by transitivity.  $\overline{OR} \parallel \overline{PS}$  since they are both perpendicular to the same line, therefore  $ORSP$  is a parallelogram, specifically a rectangle.  $OP = RS$ , and since  $G$  is the midpoint of  $\overline{OP}$  and  $T$  is the midpoint of  $\overline{RS}$  it follows that  $GP = TS$ . Therefore  $GTSP$  is a parallelogram, more specifically a square. So one solution to help José is the following: he needs to find and mark the midpoint between the Pine and the Oak. Then starting at the pine tree he should walk toward the marker while counting his steps, then make a  $90^\circ$  turn to the right and pace off the same number of paces. The treasure is at this point.

We can provide a proof of our result by coordinate geometry or by transformations.

### 1. Solution by coordinate geometry:

José should be happy now with his treasure, but in the preceding argument we made a fairly big assumption, so our conclusion is only as strong as our assumptions. Using coordinate geometry we can develop a proof of the treasure's location without making such assumptions.

- Pick convenient coordinate axes. The pine and oak trees are the only clear references. Let the pine tree be the origin and the oak tree some point on the  $y$ -axis  $(0, a)$ . The gallows are in an unknown position, say  $(x, y)$ .
- Calculate the position of Spike 2 ( $S$ ). Rotating the gallows position  $-90^\circ$  about the pine tree gives the coordinate of  $S$  as  $(y, -x)$ .

- Calculate the position of Spike1 (R). Rotating the gallows position  $90^\circ$  about the oak tree will take a little more effort. If the oak tree were the origin then the rotation of  $90^\circ$  would be simple. So let's reduce our task to a more simple task. Translate the entire picture,  $T_{(0, -a)}$ . This will place the oak tree on the origin. Rotate the translated gallows  $(x, y - a)$   $90^\circ$  about the origin to  $(-y + a, x)$ . Now translate the picture  $T_{(0, a)}$  and the picture is back where it began. The position of R is now  $(-y + a, x + a)$ .
- Our last task is to calculate where the treasure is located.

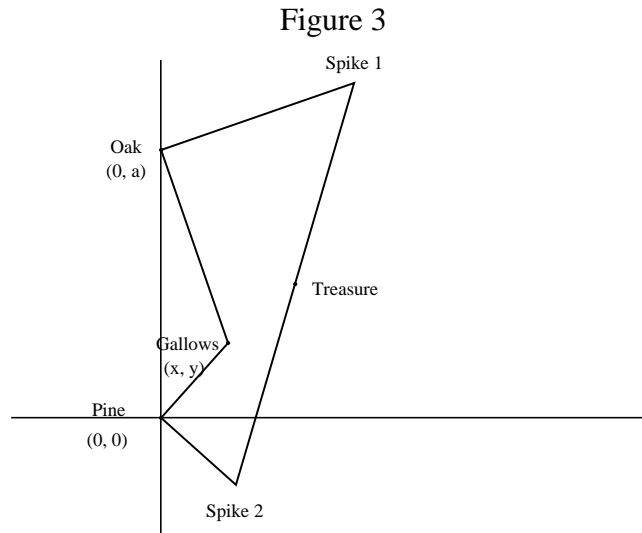
Use the midpoint formula to calculate the position of the treasure halfway between the spikes.

Spike 1: R  $(-y + a, x + a)$

Spike 2: S  $(y, -x)$

Treasure: T  $(a/2, a/2)$

Coordinate geometry proves that the position of the treasure is invariant with respect to the gallows.



## 2. Explanation by Isometries:

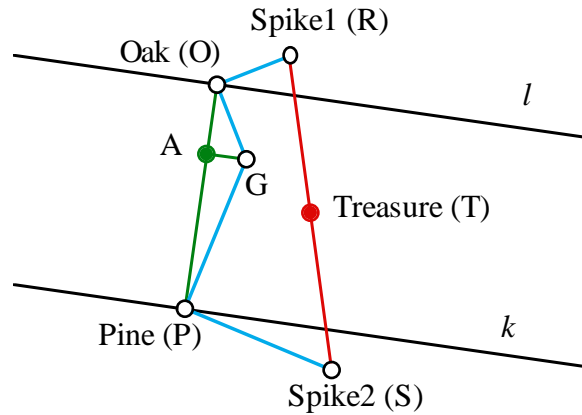
So far the explanations have given a solution, but they haven't given us much insight as to why the location of the treasure is independent of the position of the gallows. *Sketchpad* can assist in the explanation using transformations.

### 4.7.2 Demonstration. The Buried Treasure Problem using Sketchpad.

The exact position of the gallows is unknown, therefore we indicate the position of the Gallows by the letter  $G$  and make no more assumptions about its position. Construct the segment joining the Oak tree ( $O$ ) and Pine tree ( $P$ ). Construct lines  $l$  and  $k$  perpendicular to  $\overline{OP}$  passing through  $O$  and  $P$  respectively. Lines  $l$  and  $k$  are parallel to each other. Construct  $\overline{GA}$  as the altitude of the  $\triangle POG$ . By the instructions given in the map, construct the positions of the spikes ( $R$  and  $S$ ), and the treasure ( $T$ ). Hide all unnecessary lines and points. (Figure 4)



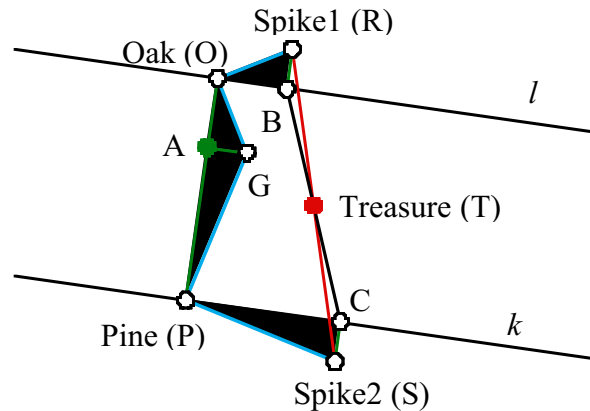
Figure 4



In the coordinate proof the spike positions were found by rotating the position of the gallows about the trees. We will use this technique again in this proof. Rotate  $\triangle OAG$   $90^\circ$  about  $O$ , forming  $\triangle OBR$ . Rotate  $\triangle PAG$   $-90^\circ$  about  $P$ , forming  $\triangle PCS$ . It is simple to show  $B$  lies on  $l$  and  $C$  lies on  $k$ . Since isometries preserve distance the following congruencies hold:  $\overline{OA} \cong \overline{OB}$ ;  $\overline{OA} \cong \overline{OC}$ , and by transitivity  $\overline{OB} \cong \overline{OC}$ . Since  $\overline{OB} \parallel \overline{OC}$ ,  $\angle BRT \cong \angle CST$ . By SAS  $\triangle RBT \cong$

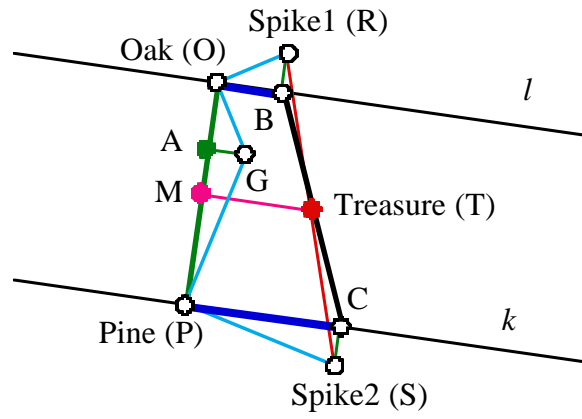
$\triangle SCT$ . From this we can conclude  $B, T, C$  are collinear,  $T$  is the midpoint of  $\overline{BC}$  and therefore equidistant from  $l$  and  $k$ . (See **Figure 5**).

Figure 5  $\rightarrow$



With  $T$  established as the midpoint of  $\overline{BC}$ , we will change our focus to the trapezoid  $OBCP$  (See Figure 6). Naming  $M$  the midpoint of  $\overline{OP}$ , yields the median  $\overline{MT}$ . The length of the median is the average of the two bases, thus  $MT = \frac{1}{2}(OB + PC)$ . But by the original rotation we know that  $OB + PC = OA + AP = OP$ ; thus  $MT = \frac{1}{2}OP$ . From this we can conclude that  $\triangle PMT$  is an isosceles right triangle.

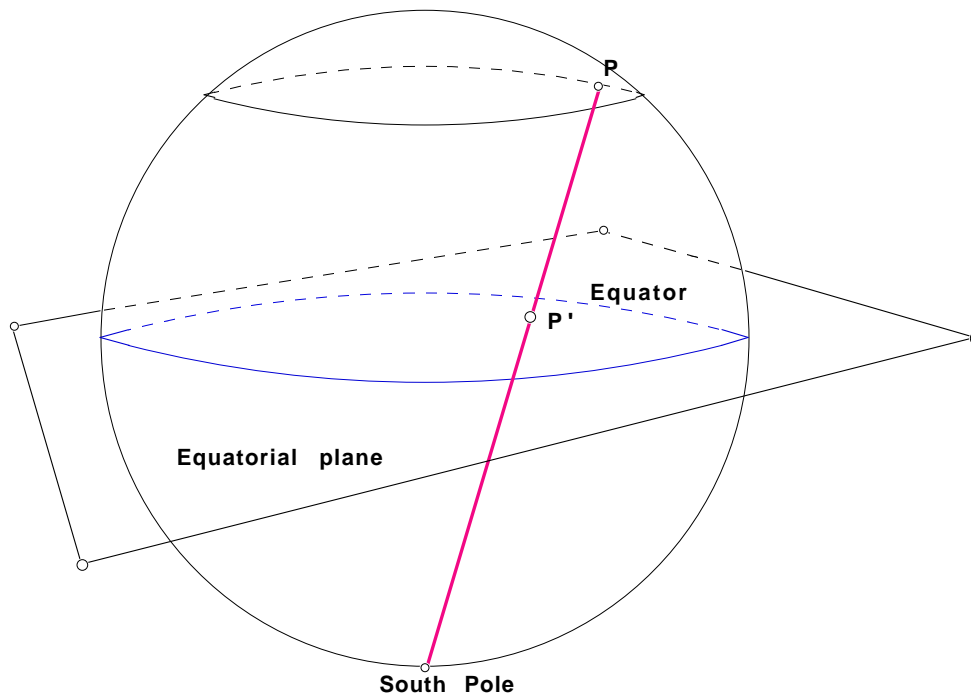
Figure 6



**End of Demonstration 4.7.2.**

**4.8 STEREOGRAPHIC PROJECTION.** In all the previous discussions the geometric transformation has mapped one model of a geometry onto the *same* model. But in map-making, for instance, the problem is to map the sphere model to a different model, in fact to a model realized as some geometry realized in the plane. One very important example of this is the transformation known as *Stereographic Projection*. We shall see this plays also a crucial role in describing the geometric transformation taking the line model of hyperbolic geometry in terms of lines and planes inside a cone in 3-space to the Poincaré model **D**.

To construct the stereographic projection of the sphere onto the plane, first draw the equatorial plane - this will serve as the plane onto which the sphere is mapped. Now take any point  $P$  on the sphere other than the South Pole and draw the ray starting at the South Pole and passing through  $P$ . Label by  $P'$  the point of intersection of this ray with the equatorial plane. For clarity in the figure below the ray has been drawn as the line segment joining the South Pole and  $P$ .



Stereographic projection is the mapping  $P \rightarrow P'$  from the sphere to the equatorial plane. It has a number of important **properties**:

1. When  $P$  lies on the equator, then  $P = P'$  so the image of the equator is itself. More precisely, the equator is left fixed by the transformation  $P \rightarrow P'$ . For convenience, let's agree to call this circle the *equatorial circle*.

2. When  $P$  lies in the Northern hemisphere then  $P'$  lies inside the equatorial circle, while if  $P$  lies in the Southern hemisphere,  $P'$  lies outside the equatorial circle.
3. Since the ray passing through the South Pole and  $P$  approaches the tangent line to the sphere at the South Pole, and so becomes parallel to the equatorial plane, as  $P$  approaches the South Pole, the image of the South Pole under stereographic projection is identified with infinity in the equatorial plane.
4. There is a 1-1 correspondence between the equatorial plane and the set of all points on the sphere excluding the South Pole.
5. The image of any line of longitude, *i.e.*, any great circle passing through the North and South Poles, is a straight line passing through the center of the equatorial circle. Conversely, the pre-image of any straight line through the center of the equatorial circle is a line of longitude on the sphere.
6. The image of any line of latitude on the sphere is a circle in the equatorial plane concentric to the equatorial circle.
7. The image of any great circle on the sphere is a circle in the equatorial plane. Now every great circle intersects the equator at diametrically opposite points on the equator. On the other hand, the points on the equator are fixed by stereographic projection, so we see that the image of any great circle on the sphere is a circle in the equatorial plane passing through diametrically opposite points on the equatorial circle.
8. Stereographic projection is *conformal* in the sense that it preserves angle measure. In other words, if the angle between the tangents at the point of intersection of two great circles is  $\theta$ , then the angle between the tangents at the points of intersection of the images of these great circles is again  $\theta$ .

Many books develop the properties of stereographic projection listed above by using the idea of inversion in 3-space. These same properties can, however, be established algebraically. This is what we'll do at this juncture because it brings in results learned earlier in calculus courses. Let  $\Sigma$  be the sphere in 3-space centered at the origin having radius 1. The points on  $\Sigma$  can be described by

$$(\xi, \eta, \zeta), \quad \xi^2 + \eta^2 + \zeta^2 = 1,$$

so in the figure above, let  $P = P(\xi, \eta, \zeta)$  and let  $P' = P'(x, y)$  be its image in the equatorial plane under stereographic transformation where the center of the equatorial circle is taken as the origin. In particular, the equation of the equatorial circle is  $x^2 + y^2 = 1$ . To determine the relation between  $(\xi, \eta, \zeta)$  and  $(x, y)$  we use similar triangles to show that

$$(A) \quad \Pi : P(\xi, \eta, \zeta) \rightarrow P'(x, y), \quad x = \frac{\xi}{1 + \zeta}, \quad y = \frac{\eta}{1 + \zeta}.$$

This is the algebraic formulation of stereographic projection. Since  $\xi^2 + \eta^2 + \zeta^2 = 1$ , the coordinates of  $P'(x, y)$  satisfy the relation

$$(B) \quad x^2 + y^2 = \frac{1 - \zeta^2}{(1 + \zeta)^2} = \frac{1 - \zeta}{1 + \zeta}.$$

As illustration, consider the case first of the North Pole  $P = (0, 0, 1)$ . Under stereographic projection  $\Pi$  maps  $P = (0, 0, 1)$  maps to  $P' = (0, 0)$  in the equatorial plane, *i.e.*, to the origin in the equatorial plane. By contrast, the South Pole is the point  $P = (0, 0, -1)$  and it is the only point of the sphere with  $\zeta = -1$ . Thus the South Pole is the only point on  $\Sigma$  for which the denominator  $1 + \zeta = 0$ . Thus the south Pole maps to infinity in the equatorial plane, and it is the only point on  $\Sigma$  which does so. That  $P(\xi, \eta, \zeta) \rightarrow P'(x, y)$  is a 1-1 mapping from  $\Sigma \setminus (0, 0, -1)$  onto the equatorial plane can also be shown solving the equations

$$x = \frac{\xi}{1 + \zeta}, \quad y = \frac{\eta}{1 + \zeta}$$

given a point  $(\xi, \eta, \zeta)$  in  $\Sigma \setminus (0, 0, -1)$  or a point  $(x, y)$  in the equatorial plane.

Now let's turn to the important question of what  $\Pi$  does to circles on  $\Sigma$ . Every such circle is the intersection with  $\Sigma$  of a plane; for instance, a great circle is the intersection of  $\Sigma$  and a plane through the origin. In calculus you learned that a plane is given by the equation

$$(C) \quad A\xi + B\eta + C\zeta = D$$

where the vector  $(A, B, C)$  is the normal to the plane and  $\frac{D}{\sqrt{A^2 + B^2 + C^2}}$  is the distance of the plane from the origin. The simplest case is that of a line of longitude. Algebraically, this is the intersection of  $\Sigma$  with a *vertical* plane through the origin, so the normal lies in the  $(\xi, \eta)$ -plane meaning that  $C = D = 0$  in the equation above. Thus a line of longitude is the set of points  $(\xi, \eta, \zeta)$  such that

$$A\xi + B\eta = 0, \quad \xi^2 + \eta^2 + \zeta^2 = 1.$$

The image of any such point under  $\Pi$  is the set of points  $(x, y)$  in the equatorial plane such that  $Ax + By = 0$ , which is the general equation of a straight line passing through the origin. Conversely, given any straight line  $l$  in the equatorial plane, it will be given by  $Ax + By = 0$  for some choice of constants  $A, B$ . So  $l$  will be the image of the great circle defined by the plane  $A\xi + B\eta = 0$ . This shows that there is a 1-1 correspondence between lines of longitude and straight lines through the center of the equatorial circle, proving property **5** above.

The image of a line of latitude is easily determined also since a line of latitude is the intersection of  $\Sigma$  with a *horizontal* plane, *i.e.*, a plane  $\zeta = D$  with  $-1 < D < 1$ . But then, by the general relation (B) the image of the line of latitude determined by the plane  $\zeta = D$  consists of all points  $(x, y)$  in the equatorial plane such that

$$x^2 + y^2 = \frac{1-D}{1+D}.$$

This is the equation of a circle centered at the origin and radius  $\sqrt{(1-D)/(1+D)}$ ; as  $D$  varies over the range  $-1 < D < 1$ , this describes the family of all circles centered at the origin. So  $\Pi$  defines a 1-1 mapping of the lines of latitude onto the family of all circles concentric with the equatorial circle.

The proof of property 7 is a little more tricky. Consider first the case of a plane passing through the points  $(0, \pm 1, 0)$  on  $\Sigma$ ; we could think of these as being the East and West ‘Poles’. Also, the plane need not be vertical because otherwise its intersection with  $\Sigma$  would be a line of longitude dealt with earlier in property 6. Thus we are led to considering a great circle determined by the plane

$$\zeta = \xi \tan \theta,$$

where  $\theta$  is fixed,  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ ; in fact,  $\theta$  is the angle between the plane and the  $(\xi, \eta)$ -plane. By relations (A) and (B), the points  $(x, y)$  in the image of the intersection with  $\Sigma$  of the plane  $\zeta = \xi \tan \theta$  will satisfy the equations

$$x = \frac{\xi}{1 + \xi \tan \theta}, \quad x^2 + y^2 = \frac{1 - \zeta}{1 + \zeta} = \frac{1 - \xi \tan \theta}{1 + \xi \tan \theta}.$$

After eliminating  $\xi$  from these equations we see that the image point  $(x, y)$  satisfies the equation

$$x^2 + y^2 = 1 - 2x \tan \theta.$$

In other words, the image of the great circle determined by the plane  $\zeta = \xi \tan \theta$  is the circle

$$(x + \tan \theta)^2 + y^2 = 1 + (\tan \theta)^2 = (\sec \theta)^2$$

which is the circle centered at  $(-\tan \theta, 0)$  having radius  $1/\cos \theta$ . As problem 7 in Assignment 6 shows, this is a circle passing through diametrically opposite points of the circle  $x^2 + y^2 = 1$ ; in fact, it passes through the points  $y = \pm 1$  which are the image of the points of intersection of the great circles determined by the plane  $\zeta = \xi \tan \theta$  and the equator in  $\Sigma$ .

But how do we deal with a more general great circle that is not a line of longitude and does not pass through the East and West Poles? The fundamental idea we’ll use is that a rotation of the sphere about the  $\zeta$ -axis through an angle  $\phi$  will fix the  $\zeta$ -coordinate of a

point  $P(\xi, \eta, \zeta)$  on  $\Sigma$  while rotating the  $\xi, \eta$ -coordinates, but it will also rotate the  $x, y$ -coordinates of the image  $P'(x, y)$  by the same angle  $\phi$ . So the effect of rotating a great circle is to rotate its image under stereographic projection. Since a rotation is an isometry, it maps a circle to a circle. Hence the image of any great circle is a circle. Let's do the details.

**4.8.1 Theorem.** Under the rotation  $\rho_{O, \phi}$  about the origin the point  $(\xi, \eta)$  is mapped to the point  $(\xi', \eta') = \rho_{O, \phi}(\xi, \eta)$  where

$$\xi' = \xi \cos \phi - \eta \sin \phi, \quad \eta' = \xi \sin \phi + \eta \cos \phi.$$

More generally, the point  $(\xi, \eta, \zeta)$  is mapped to the point  $(\xi', \eta', \zeta)$ .

Under  $\rho_{O, \phi}$  the plane  $\zeta = \xi \tan \theta$  is mapped to the plane  $\zeta = (\xi \cos \phi + \eta \sin \phi) \tan \theta$ . The angle between this plane and the  $(\xi, \eta)$ -plane is again  $\theta$  and the intersection of the plane with  $\Sigma$  is a great circle passing through the equator at the points

$$(-\sin \phi, \cos \phi, 0), \quad (\sin \phi, -\cos \phi, 0).$$

Now by (A), the point  $(\xi', \eta', \zeta)$  is mapped to  $(x', y')$  where

$$x' = x \cos \phi - y \sin \phi, \quad y' = x \sin \phi + y \cos \phi.$$

Consequently, stereographic projection commutes with the rotation  $\rho_{O, \phi}$  in the sense that

$$(D) \quad \Pi \circ \rho_{O, \phi} = \rho_{O, \phi} \circ \Pi.$$

Since the isometry  $\rho_{O, \phi}$  will map circles to circles, we obtain the following result, completing the proof of property 7 listed above.

**4.8.2 Theorem.** Stereographic projection maps the great circle determined by the rotated plane  $\zeta = (\xi \cos \phi + \eta \sin \phi) \tan \theta$  to the circle in the equatorial plane obtained after rotation by  $\rho_{O, \phi}$  of the image of the great circle determined by the plane  $\zeta = \xi \tan \theta$ .



The general result of property 8 can be established using similar transformation ideas to those in the proof of Theorem 4.8.2.