

Chapter 1

GEOMETRY: Making a Start

1.1 INTRODUCTION. The focus of geometry continues to evolve with time. The renewed emphasis on geometry today is a response to the realization that visualization, problem-solving and deductive reasoning must be a part of everyone's education. Deductive reasoning has long been an integral part of geometry, but the introduction in recent years of inexpensive dynamic geometry software programs has added visualization and individual exploration to the study of geometry. All the constructions underlying Euclidean plane geometry can now be made accurately and conveniently. The dynamic nature of the construction process means that many possibilities can be considered, thereby encouraging exploration of a given problem or the formulation of conjectures. Thus geometry is ideally suited to the development of visualization and problem solving skills as well as deductive reasoning skills. Geometry itself hasn't changed: technology has simply added a powerful new tool for use while studying geometry.

So what is geometry? Meaning literally "earth measure", geometry began several thousand years ago for strictly utilitarian purposes in agriculture and building construction. The explicit 3-4-5 example of the Pythagorean Theorem, for instance, was used by the Egyptians in determining a square corner for a field or the base of a pyramid long before the theorem as we know it was established. But from the sixth through the fourth centuries BC, Greek scholars transformed empirical and quantitative geometry into a logically ordered body of knowledge. They sought irrefutable proof of abstract geometric truths, culminating in Euclid's Elements published around 300 BC. Euclid's treatment of the subject has had an enormous influence on mathematics ever since, so much so that deductive reasoning is the method of mathematical inquiry today. In fact, this is often interpreted as meaning "geometry is 2-column proofs". In other words geometry is a formal axiomatic structure – typically the axioms of Euclidean plane geometry - and one objective of this course is to develop the axiomatic approach to various geometries, including plane geometry. This is a very important, though limited, interpretation of the need to study geometry, as there is more to learn from geometry than formal axiomatic structure. Successful problem solving requires a deep knowledge of a large body of geometry

and of different geometric techniques, whether or not these are acquired by emphasizing the ‘proving’ of theorems.

Evidence of geometry is found in all cultures. Geometric patterns have always been used to decorate buildings, utensils and weapons, reflecting the fact that geometry underlies the creation of design and structures. Patterns are visually appealing because they often contain some symmetry or sense of proportion. Symmetries are found throughout history, from dinosaur tracks to tire tracks. Buildings remain standing due to the rigidity of their triangular structures. Interest in the faithful representation of a three dimensional scene as a flat two-dimensional picture has led artists to study perspective. In turn perspective drawing led to the introduction of projective geometry, a different geometry from the plane geometry of Euclid. The need for better navigation as trading distances increased along with an ever more sophisticated understanding of astronomy led to the study of spherical geometry. But it wasn't until the 19th century, as a result of a study examining the role of Euclid's parallel postulate, that geometry came to represent the study of the geometry of surfaces, whether flat or curved. Finally, in the 20th century this view of geometry turned out to be a vital component of Einstein's theory of relativity. Thus through practical, artistic and theoretical demands, geometry evolved from the flat geometry of Euclid describing one's immediate neighborhood, to spherical geometry describing the world, and finally to the geometry needed for an understanding of the universe.

The most important contribution to this evolution was the linking of algebra and geometry in coordinate geometry. The combination meant that algebraic methods could be added to the synthetic methods of Euclid. It also allowed the use of calculus as well as trigonometry. The use of calculus in turn allowed geometric ideas to be used in real world problems as different as tossing a ball and understanding soap bubbles. The introduction of algebra also led eventually to an additional way of thinking of congruence and similarity in terms of groups of transformations. This group structure then provides the connection between geometry and the symmetries found in geometric decorations.

But what is the link with the plane geometry taught in high school which traditionally has been the study of congruent or similar triangles as well as properties of circles? Now congruence is the study of properties of figures whose size does not change when the figures are moved about the plane, while similarity studies properties of figures whose shape does not change. For instance, a pattern in wallpaper or in a floor covering is likely to be interesting when the pattern does not change under some reflection or rotation. Furthermore, the physical problem of actually papering a wall or laying a tile floor is made possible because the pattern repeats in directions parallel to the sides of the wall or floor, and thereby does not change under translations in two directions. In this way geometry becomes a study of properties that do not change under a family of transformations. Different families determine different geometries or

different properties. The approach to geometry described above is known as Klein's Erlanger Program because it was introduced by Felix Klein in Erlangen, Germany, in 1872.

This course will develop all of these ideas, showing how geometry and geometric ideas are a part of everyone's life and experiences whether in the classroom, home, or workplace. To this is added one powerful new ingredient, technology. The software to be used is Geometer's Sketchpad. It will be available on the machines in this lab and in another lab on campus. Copies of the software can also be purchased for use on your own machines for approximately \$45 (IBM or Macintosh). If you are 'uncertain' of your computer skills, don't be concerned - one of the objectives of this course will be to develop computer skills. There's no better way of doing this than by exploring geometry at the same time.

In the first chapter of the course notes we will cover a variety of geometric topics in order to illustrate the many features of Sketchpad. The four subsequent chapters cover the topics of Euclidean Geometry, Non-Euclidean Geometry, Transformations, and Inversion. Here we will use Sketchpad to discover results and explore geometry. However, the goal is not only to study some interesting topics and results, but to also give "proof" as to why the results are valid and to use Sketchpad as a part of the problem solving process.

1.2 EUCLID'S ELEMENTS. The *Elements* of Euclid were written around 300 BC. As Eves says in the opening chapter of his 'College Geometry' book,

"this treatise by Euclid is rightfully regarded as the first great landmark in the history of mathematical thought and organization. No work, except the Bible, has been more widely used, edited, or studied. For more than two millennia it has dominated all teaching of geometry, and over a thousand editions of it have appeared since the first one was printed in 1482. ... It is no detraction that Euclid's work is largely a compilation of works of predecessors, for its chief merit lies precisely in the consummate skill with which the propositions were selected and arranged in a logical sequence ... following from a small handful of initial assumptions. Nor is it a detraction that ... modern criticism has revealed certain defects in the structure of the work."

The *Elements* is a collection of thirteen books. Of these, the first six may be categorized as dealing respectively with triangles, rectangles, circles, polygons, proportion and similarity. The next four deal with the theory of numbers. Book XI is an introduction to solid geometry, while XII deals with pyramids, cones and cylinders. The last book is concerned with the five regular solids. Book I begins with twenty three definitions in which Euclid attempts to define the notion of 'point', 'line', 'circle' *etc.* Then the fundamental idea is that all subsequent theorems – or Propositions as Euclid calls them – should be deduced logically from an initial set of assumptions. In all, Euclid proves 465 such propositions in the *Elements*. These are listed in

detail in many texts and not surprisingly in this age of technology there are several web-sites devoted to them. For instance,

<http://aleph0.clarku.edu/~djoyce/java/Geometry/Geometry.html>

is a very interesting attempt at putting Euclid's *Elements* on-line using some very clever Java applets to allow real time manipulation of figures; it also contains links to other similar web-sites. The web-site

<http://thales.vismath.org/euclid/>

is a very ambitious one; it contains a number of interesting discussions of the *Elements*.

Any initial set of assumptions should be as self-evident as possible and as few as possible so that if one accepts them, then one can believe everything that follows logically from them. In the *Elements* Euclid introduces two kinds of assumptions:

COMMON NOTIONS:

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

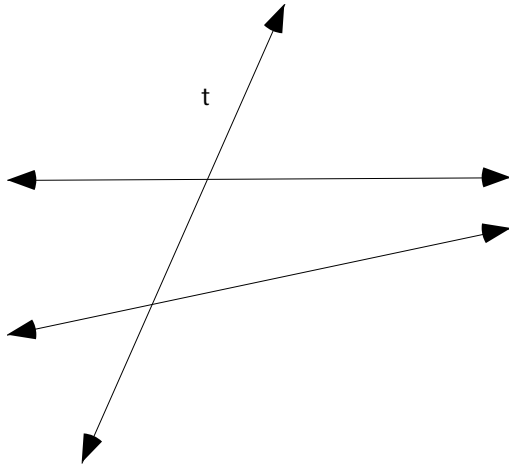
POSTULATES: Let the following be postulated.

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines if produced indefinitely, meet on that side on which are the angles less than two right angles.

Today we usually refer to all such assumptions as *axioms*. The common notions are surely self-evident since we use them all the time in many contexts not just in plane geometry – perhaps that's why Euclid distinguished them from the five postulates which are more geometric in character. The first four of these postulates too seem self-evident; one surely needs these constructions and the notion of perpendicularity in plane geometry. The Fifth postulate is of a more technical nature, however. To understand what it is saying we need the notion of parallel lines.

1.2.1 Definition. Two straight lines in a plane are said to be *parallel* if they do not intersect, *i.e.*, do not meet.

The Fifth postulate, therefore, means that straight lines in the plane are not parallel when there is a transversal t such that the sum ($+$) of the interior angles on one side is less than the sum of two right angles; in fact, the postulate states that the lines must meet on this side.



The figure above makes this clear. The need to assume this property, rather than showing that it is a consequence of more basic assumptions, was controversial even in Euclid's time. He himself evidently felt reluctant to use the Fifth postulate, since it is not used in any of the proofs of the first twenty-eight propositions in Book I. Thus one basic question from the time of Euclid was to decide if the Fifth Postulate is independent of the Common Notions and the first four Postulates or whether it could be deduced from them.

Attempts to deduce the Fifth postulate from the Common Notions and other postulates led to many statements logically equivalent to it. One of the best known is

1.2.2 Playfair's Axiom: Through a given point, not on a given line, exactly one line can be drawn parallel to the given line.

Its equivalence to the Fifth Postulate will be discussed in detail in Chapter 2. Thus the Fifth postulate would be a consequence of the Common notions and first four postulates if it could be shown that neither

ALTERNATIVE A: through a given point not on a given line, no line can be drawn parallel to the given line, nor

ALTERNATIVE B: through a given point not on a given line, more than one line can be drawn parallel to the given line

is possible once the five Common notions and first four postulates are accepted as axioms. Surprisingly, the first of these alternatives does occur in a geometry that was familiar already to the Greeks, replacing the plane by a sphere. On the surface of the earth, considered as a sphere, a great circle is the curve formed by the intersection of the earth's surface with a plane passing through the center of the earth. The arc between any two points on a great circle is the shortest distance between those two points. Great circles thus play the role of 'straight lines' on the sphere and arcs of great circles play the role of line segments. In practical terms, arcs of great circles are the most efficient paths for an airplane to fly in the absence of mountains or for a ship to follow in open water. Hence, if we interpret 'point' as having its usual meaning on a sphere and 'straight line' to mean great circle, then the resulting geometry satisfies Alternative A because two great circles must always intersect (why?). Notice that in this geometry 'straight lines' are finite in length though they can still be continued indefinitely as required by the second Postulate.

This still leaves open the possibility of Alternative B. In other words, there might be geometry in which Alternative B occurs, and hence a geometry in which Alternative B is a legitimate logical substitute for Playfair's axiom. If so, the familiar results of Euclidean geometry whose proofs rely on the Fifth postulate would not necessarily remain true in this geometry. In the early 19th century Gauss, Lobachevsky, and Bolyai showed that there indeed exists such a logically reasonable geometry – what we now call hyperbolic geometry. It is based on Alternative B together with the five common notions and first four postulates of Euclid. Towards the end of the 19th century simple 'models' of hyperbolic plane geometry were given by Poincaré and others in terms of two and three dimensional Euclidean geometry. As a result of this discovery of hyperbolic geometry, the mathematical world has been radically changed since Alternative B appears to run counter to all prior experiences. Thus Euclidean plane geometry is only one possible geometry - the one that follows by adopting the Fifth Postulate as an axiom. For this reason, the Fifth Postulate is often referred to as the Euclidean parallel postulate, and these notes will continue this convention. Some interesting consequences of the Euclidean Parallel postulate beyond those studied in high school will be developed in Chapter 2.

The first three postulates of Euclid reflect the growth of formal geometry from practical constructions – figures constructed from line segments and circles – and the same can be said for many of the subsequent propositions proved by Euclid. We will see that software will allow constructions to be made that Euclid could only describe in words or that previously one could draw only in a rudimentary fashion using ruler and compass. This software will provide a rapid

and accurate means for constructing line-segments, lines, and circles, as well as constructions based upon these objects. It will enable us to construct accurate geometric configurations that in turn can be altered to new figures having the same construction constraints. This ability to drag the figure about has been available only within the past decade. It allows a student to carry out geometric experiments quickly, producing accurate sketches from which ‘conjectures’ can be made. These conjectures can then be in turn verified in whatever manner is deemed appropriate.

The *Geometer’s Sketchpad* referred to in these notes, is such a software program. It provides accurate constructions and measures of geometric configurations of points, line segments, circles, etc. and it has the ability to replay a given construction. The software can be used to provide visually compelling evidence of invariance properties such as concurrence of lines, the collinearity of points, or the ratios of particular measurements. In addition, Sketchpad allows translations, rotations, reflections and dilations of geometric constructions to be made either singly or recursively, permitting the study of transformations in a visually compelling way as will be seen in Chapters 2 and 4. Because the two-dimensional models of hyperbolic geometry – the so called *Poincaré disk* and *upper half-plane* models - make extensive use of circles and arcs of circles, Geometer’s Sketchpad is also particularly well-adapted to developing hyperbolic plane geometry as we shall see in Chapters 3 and 5.

1.3 GEOMETER’S SKETCHPAD. Successful use of any software requires a good working knowledge of its features and its possibilities. One objective of this course is the development of that working knowledge.

Basic geometric figures are constructed using the drawing tools in the toolbox and the dynamic aspect of Sketchpad can be exploited by using the selection arrow to drag any figure that has been constructed. The Measure menu allows us to measure properties of a figure. With the Edit and Display menus labels can be added to figures, and those figures can be animated. Using scripts we also can replay complex geometric constructions in a single step. To start with we will use some of the more basic tools of Sketchpad - a more extensive listing is given in Appendix A.

General Instructions: The set of squares along the left-hand side of the screen comprises the toolbox. The tools in the toolbox are (from top to bottom):

- **Selection Arrow tools:** Press and hold down the mouse clicker for Rotate and Dilate tools.
- **Point tool:** Creates points.
- **Compass (Circle) tool:** Creates circles

- **Straightedge (Segment) tool:** Press and hold down the mouse clicker for Ray and Line tools.
- **Text:** Click on object to display or hide label. Drag label to reposition. Double click on label, measure or caption to edit or change style. Drag in blank area to create caption.
- **Information tool:** Press to display menu of current selections. Choose item from menu for information. Click on object in sketch for info balloon. Double click for more information.

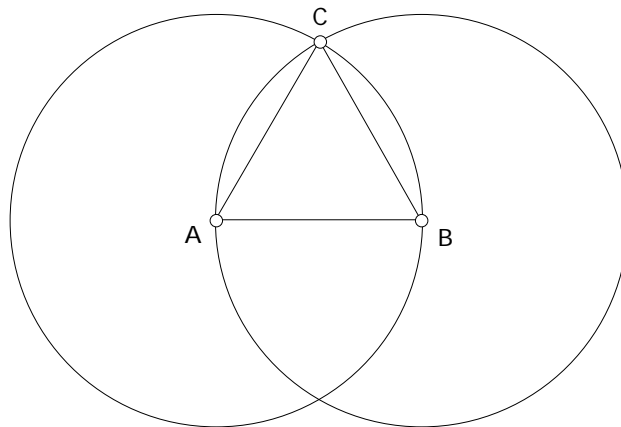
These notes contain several **Demonstrations**. In a **Demonstration**, a problem or task is proposed and the solution to the problem or task is described in the body of the **Demonstration**.

To get started using Sketchpad let's consider this **Demonstration**.

1.3.1 **Demonstration:** Construct an equilateral triangle using Geometer's Sketchpad.

In other words, using Sketchpad construct a triangle that remains equilateral no matter how we drag each of the vertices around the sketch using the Arrow tool. Here are the steps for one of several possible constructions.

- Open a new sketch. To create a new sketch, select "New Sketch" from under the **File** menu. Using the Segment tool, draw a line segment, and label its endpoints A , B . This defines one side of the equilateral triangle. The idea for our construction will be to construct the remaining sides so that they have length equal to that of \overline{AB} . To accomplish this we construct a circle passing through A with radius \overline{AB} as well as a circle passing through B with the same radius. Either point of intersection of these circles can then form the third vertex C of an equilateral triangle ABC .
- Select vertices A and B while holding down the Shift key. Select "Circle By Center And Point" from under the **Construct** menu. Note that the order in which the points A and B are selected determines which is the center of the circle and which point lies on the circle. Repeat to construct a circle centered at the other endpoint.
- Using the Select arrow, and while holding down the Shift key, select the two circles. Select "Point At Intersection" from under the **Construct** menu. Using the Text tool, label the point C .
- To finish ABC , use the Segment tool to construct \overline{AC} and \overline{CB} . The resulting figure should look similar to



Hide everything in this figure except the required equilateral triangle by simultaneously selecting the undesired objects. Then choose “Hide Objects” from the **Display Menu**. You may click on objects individually with the Arrow Tool or you may use the Arrow Tool to drag over an area and select more than one object at once. If you selected too many objects while holding the shift key you can deselect the unwanted objects with the Arrow Tool again. Drag either A or B to verify that ABC remains equilateral. Does dragging vertex C have the same effect as dragging vertex A ? The answer should be no. This is due to the fact that vertex C is not a *free point* because it was constructed *from* A and B . The vertex A is a free point so A might be thought of as a *independent* variable and C as a *dependent* variable. To save your figure select “Save” from under the **File** menu. The convention is to save sketches with the file extension .gsp.

End of Demonstration 1.3.1.

We can use measuring features of Sketchpad to confirm that we do have an equilateral triangle. Select the three sides of the equilateral triangle then select “Length” from the **Measure** menu. The lengths of the three segments should appear in the corner of your sketch. At this time you should notice the various other quantities that can be measured. Select Show Balloons from the **Help** menu to provide details on what needs to be selected to make the various measurements.

1.3.2 Exercise. Using Sketchpad, construct each of the following figures so that the figure retains its defining property when a free point on the figure is dragged:

- a) a rectangle, given perpendicular segments \overline{AB} and \overline{AC} ;
- b) a parallelogram, given two segments \overline{AB} and \overline{AC} with A , B , and C free points;

- c) a rhombus, given two segments \overline{AB} and \overline{AC} with $\overline{AB} \cong \overline{AC}$;
- d) a 30-60-90 triangle, given line segment \overline{AB} as the hypotenuse of the triangle.

1.4 GETTING STARTED. Let's review briefly some of the principal ideas typically taught in high school geometry, keeping in mind the role of the Euclidean parallel postulate and the question of how one might incorporate the use of dynamic geometric. Many of the early propositions established by Euclid dealt with constructions which were a consequence of the first four postulates, so high school geometry often begins with the following constructions:

- construct a congruent copy of a given line segment (given angle)
- bisect a given line segment (given angle)
- construct the perpendicular bisector of a given line segment
- construct a line perpendicular to a given line through a point on the given line
- construct the perpendicular line to a given line from a point not on the given line

The **Construct** menu in Sketchpad allows us to do most of these constructions in one or two steps. If you haven't done so already, look at what is available under the **Construct** menu. It is worth noting that Euclid's constructions were originally accomplished with only a compass and straightedge. On Sketchpad this translates to using only the Circle and Segment tools. Try this!

1.4.1 Exercise: Do the constructions listed above using only the Circle and Segment tools. (You can drag, label, hide etc.)

Although Euclid's fifth postulate is needed to prove many of his later theorems, he presents 28 propositions in The Elements before using that postulate for the first time. This will be important later because all these results remain valid in a geometry in which Alternative B is assumed and all but one of these remain valid in a geometry in which Alternative A is assumed. For this reason we will make careful note of the role of the fifth postulate while continuing to recall geometric ideas typically taught in high school geometry. For instance, the familiar congruence properties of triangles can be proved without the use of the Fifth postulate. In high school these may have been taught as 'facts' rather than as theorems, but it should be remembered that they could be deduced from the first four Postulates. Recall that a triangle ABC is said to be congruent to DEF , written $ABC \cong DEF$, when there is a correspondence $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$ in which all three pairs of corresponding sides are

congruent and all three pairs of corresponding angles are congruent. To establish congruence of triangles, however, it is not necessary to establish congruence of all sides and all angles.

1.4.2 Theorem (SAS). If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

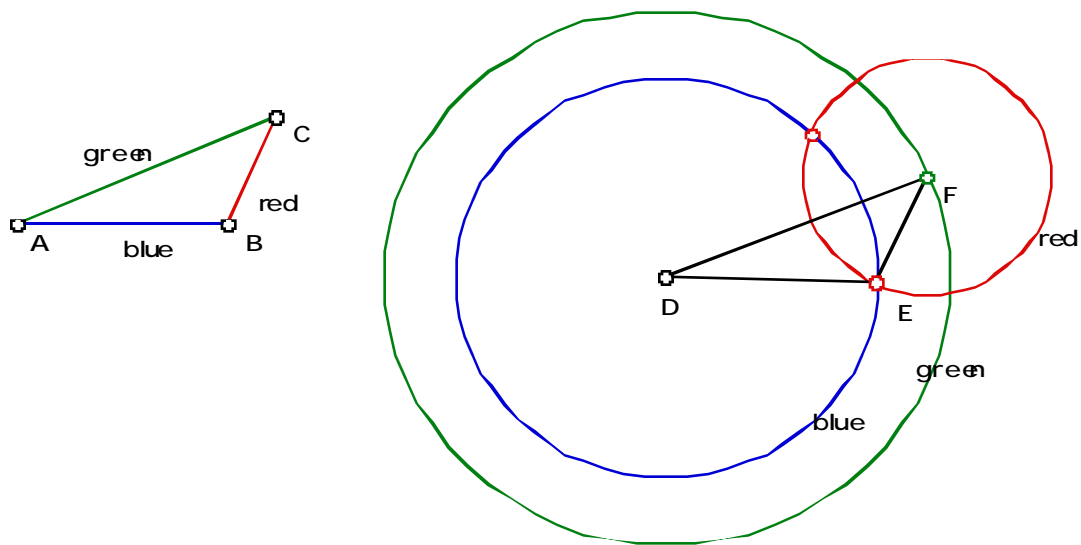
1.4.3 Theorem (ASA). If two angles and the included side of one triangle are congruent respectively to two angles and the included side of another triangle, then the two triangles are congruent.

1.4.4 Theorem (SSS). If three sides of one triangle are congruent respectively to three sides of another triangle, then the two triangles are congruent.

1.4.5. Theorem (HL). If the hypotenuse and a leg of one right triangle are congruent respectively to the hypotenuse and leg of another right triangle, then the two triangles are congruent.

These shortcuts to showing triangle congruence will be put to good use in the future. As an illustration of how we might implement them on Sketchpad consider the problem of constructing a triangle congruent to a given triangle. In more precise terms this can be formulated as follows.

1.4.5a Demonstration: Open a new sketch and construct ABC ; now construct a new triangle in this sketch congruent to ABC . Here is one solution based on the SSS shortcut.



- Open a new sketch and construct ABC using the Segment tool in the toolbar on the left of the screen. Make certain that it is the segment tool showing, not the ray or line tool. To verify that the correct tool is selected look at the toolbar, the selected tool should be purple. Now in the sketch window click down at the first vertex position, move the mouse to the second vertex and release the mouse clicker. At this same position, click down on the mouse, move the mouse to the third vertex, and release. Click down on the third vertex, and release on the first vertex. Label the vertices A , B , and C using the Text tool, re-labeling if necessary.
- Change the color of \overline{AB} to blue, \overline{BC} to red, and \overline{AC} to green. To change the color of a line segment first select the segment then select “Color” from the **Display** menu and choose the desired color.
- Construct the point D elsewhere in your sketch. Now select the point D and \overline{AB} while holding down the shift key. Using the **Construct** menu select “Circle By Center And Radius”. Change the color of the circle to blue.
- Now select the point D and \overline{AC} while holding down the shift key. Using the **Construct** menu select “Circle By Center And Radius”. Change the color of the circle to green.
- Now construct any point on the green circle and label it F . Select that F and \overline{BC} while holding down the shift key. Using the **Construct** menu select “Circle By Center And Radius”. Change the color of the circle to red.

- Construct one of the points of intersection between the red and the blue circle and label the point by E . To do this you may use the point tool to click on the intersection point directly. Alternatively, you can select both circles and using the **Construct** menu select “Point At Intersection”.
- Finally, use the segment tool to construct \overline{DE} , \overline{EF} , and \overline{DF} . By SSS $\triangle ABC$ is congruent to $\triangle DEF$. Drag the vertices of $\triangle ABC$ to observe the dynamic nature of your construction.

End of Demonstration 1.4.5a.

Two important results follow from the previous theorems about triangle congruence.

1.4.6 Theorem. In an isosceles triangle, the angles opposite the congruent sides are congruent.

1.4.7 Corollary. In an isosceles triangle, the ray bisecting the angle included by the congruent sides bisects the side opposite to this angle and is perpendicular to it.

One surprising discovery of a high school geometry course is the number of properties that the simplest of all geometric figures – a triangle – has. Recall first

Definition: Three or more lines that intersect in one point are called *concurrent lines*.

1.4.8 Theorem. The perpendicular bisectors of the sides of a triangle are concurrent at a point called the *circumcenter*.

Definition: The segment connecting the vertex of a triangle and the midpoint of its opposite side is called a *median*.

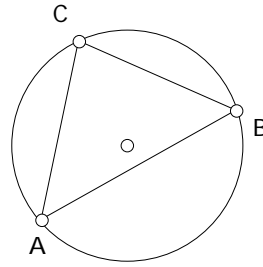
1.4.9 Theorem. The medians of a triangle are concurrent, at a point called the *centroid*. Furthermore, the centroid trisects each of the medians.

1.4.10 Theorem. The bisectors of the angles of a triangle are concurrent at a point called the *incenter*.

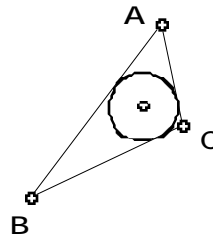
Definition: The segment connecting the vertex of a triangle and perpendicular to its opposite side is called an *altitude*.

1.4.11 Theorem. The altitudes of a triangle are concurrent at a point called the *orthocenter*.

The circumcenter is equidistant from all three of the vertices of a triangle it is also the center of the unique circle, the *circumcircle* or *circumscribing* circle, passing through the three vertices of the triangle.



The incenter is equidistant from all three sides of a triangle and so is the center of the unique circle, the *incircle* or *inscribing* circle, of a triangle.



Although the proofs of Theorems 1.4.9 and 1.4.10 (existence of the centroid and the incenter) do not require Euclid's fifth postulate, both Theorem 1.4.8 and 1.4.11 (existence of the circumcenter and the orthocenter) are dependent on the fifth postulate. In other words, no proof can be found that does not use the fifth postulate. To understand this better, you may want to try to recall the proofs of these theorems.

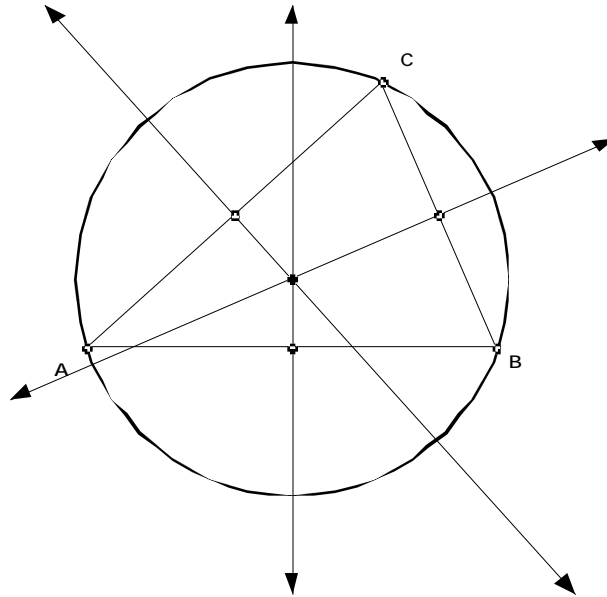
What may come as an even greater surprise is that triangles have many more properties than the ones taught in high school. In fact, there are many special points and circles associated with triangles other than the ones previously listed. The web-site

<http://www.evansville.edu/~ck6/tcenters/> lists a number of them; look also at

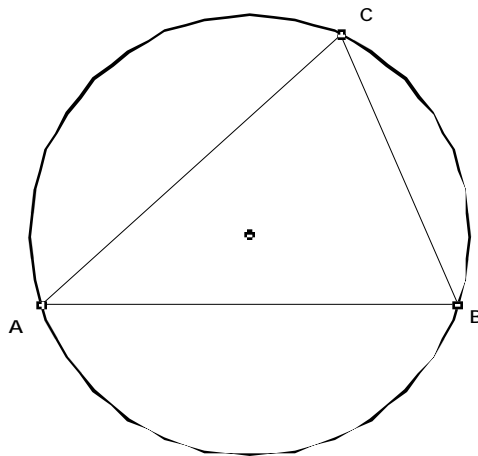
<http://www.evansville.edu/~ck6/index.html>. Sketchpad explorations will be given or suggested in subsequent sections and chapters enabling the user to discover and exhibit many of these properties. First we will look at a Sketchpad construction for the circumcircle of a triangle.

1.4.12 Demonstration: Construct the circumcircle of a given triangle.

- Open a new sketch. To construct $\triangle ABC$ use the Segment tool in the toolbar on the left of the screen. Make certain that it is the segment tool showing, not the ray or line tool. To verify that the correct tool is selected look at the toolbar, the selected tool should be purple. Now in the sketch window click down at the first vertex position, move the mouse to the second vertex and release the mouse clicker. At this same position, click down on the mouse button, move the mouse to the third vertex, and release. Click down on the third vertex, and release on the first vertex. Re-label the vertices A , B , and C using the Text tool.
- To construct a midpoint of a segment, use the Select arrow tool from the toolbar. Select a segment on screen, say \overline{AB} , by pointing the arrow at it and clicking. If done correctly, the line segment becomes highlighted by two small squares. Select “Point At Midpoint” from under the **Construct** menu. Upon releasing the mouse, the midpoint of \overline{AB} will be constructed immediately as a highlighted small circle. Repeat this procedure for the remaining two sides of $\triangle ABC$. (Note that all three midpoints can be constructed simultaneously by selecting all three sides of $\triangle ABC$ while also holding down the shift key and clicking on each of the three sides.)
- To construct a perpendicular bisector of a segment, use the Select arrow tool to select a segment and the midpoint of the segment (while holding down the Shift key). Select “Perpendicular Line” from under the **Construct** menu. Repeat this procedure for the remaining two sides of $\triangle ABC$.
- These perpendicular bisectors are concurrent at a point called the circumcenter of $\triangle ABC$, confirming visually Theorem 1.4.8.
- To identify this point as a specific point, use the arrow tool to select two of the perpendicular bisectors simultaneously while holding down the Shift key. Select “Point At Intersection” from under the **Construct** menu. In practice this means that only two perpendicular bisectors of a triangle are needed in order to find the circumcenter.
- To construct the circumcircle of a triangle, use the Select arrow to select the circumcenter and a vertex of the triangle, *in that order*. Select “Circle By Center+Point” from under the **Construct** menu. This contains all parts of the construction.



- To hide all the objects other than the triangle ABC and its circumcircle, use the Select arrow tool to select all parts of the figure *except* the triangle and the circle. Select “Hide Objects” from under the **Display** menu. The result should look similar to the following figure.



The dynamic aspect of this construction can be demonstrated by using the ‘drag’ feature. Select one of the vertices of ABC using the Select arrow and ‘drag’ the vertex to another point on the screen while holding down on the mouse button. The triangle and its circumcenter remain a triangle with a circumcenter. In other words, the construction has the *ability to replay itself*. Secondly, once this construction is completed there will be no need to repeat it every time the circumcircle of a triangle is needed because a script of the construction can be created for

use whenever a circumcircle is needed. This feature will be presented in Section 1.7, once a greater familiarity with Sketchpad's basic features has been attained.

End of Demonstration 1.4.12.

1.5 Exercises. The following problems are designed to develop a working knowledge of Sketchpad as well as provide some indication of how one can gain a good understanding of plane geometry at the same time. It is important to stress, however, that use of Sketchpad is not the only way of studying geometry, nor is it always the best way. For the exercises, in general, when a construction is called for your answer should include a description of the construction, an explanation of why the construction works and a print out of your sketches.

Exercise 1.5.1, Particular figures I: In section 1.3 a construction of an equilateral triangle starting from one side was given. This problem will expand upon those ideas.

- a) Draw a line segment and label its endpoints A and B . Construct a square having \overline{AB} as one of its sides. Describe your construction and explain why it works.
- b) Draw another line segment and label its endpoints A and B . Construct a triangle ABC having a right angle at C so that the triangle remains right-angled no matter which vertex is dragged. Explain your construction and why it works. Is the effect of dragging the same at each vertex in your construction? If not, why not?

Exercise 1.5.2, Particular figures II:

- a) Construct a line segment and label it \overline{CD} . Now construct an isosceles triangle having \overline{CD} as its base and altitude half the length of \overline{CD} . Describe your construction and explain why it works.
- b) Modify the construction so that the altitude is twice the length of \overline{CD} . Describe your construction and explain why it works.

Exercise 1.5.3, Special points of triangles: For several triangles which are not equilateral, the incenter, orthocenter, circumcenter and centroid do not coincide and are four distinct points. For an equilateral triangle, however, the incenter, orthocenter, circumcenter and centroid all coincide at a unique point we'll label by N .

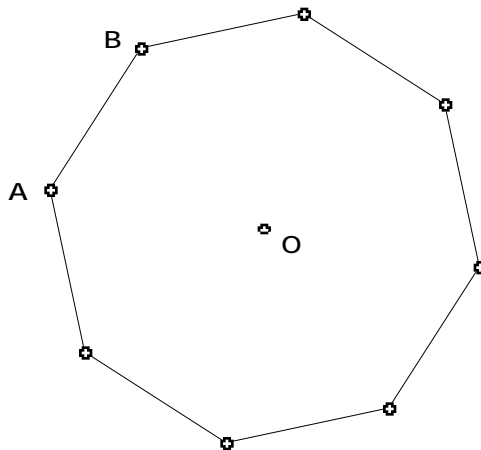
- Using Sketchpad, in a new sketch place a point and label it N . Construct an equilateral triangle ABC such that N is the common incenter, orthocenter, circumcenter and centroid of ABC . Describe your construction and explain why it works.

Exercise 1.5.4, Euclid's Constructions: Use only the segment and circle tools to construct the following objects. (You may drag, hide, and label objects.)

(a) Given a line segment \overline{AB} and a point C above \overline{AB} construct the point D on \overline{AB} so that \overline{CD} is perpendicular to \overline{AB} . We call D the *foot* of the perpendicular from C to \overline{AB} . Explain your construction.

(b) Show how to construct the bisector of a given angle ABC . Explain your construction.

Exercise 1.5.5, Regular Octagons: By definition an octagon is a polygon having eight sides; a regular octagon, as shown below, is one whose sides are all congruent and whose interior angles are all congruent:



Think of all the properties of a regular octagon you know or can derive (you may assume that the sum of the angles of a triangle is 180 degrees). For instance, one property is that all the vertices lie on a circle centered at a point O . Use this property and others to complete the following.

(a) Using Sketchpad draw two points and label them O and A , respectively. Construct a regular octagon having O as center and A as one vertex. In other words, construct an octagon by center and point.

(b) Open a new sketch and draw a line segment \overline{CD} (don't make it too long). Construct a regular octagon having \overline{CD} as one side. In other words, construct an octagon by edge.

Exercise 1.5.6, Lost Center: Open a new sketch and select two points; label them O and A . Draw the circle centered at O and passing through A . Now hide the center O of the circle. How could you recover O ? *EASY WAY:* if hiding O was the last keystroke, then “Undo hide point” can be used. Instead, devise a construction that will recover the center of the circle - in other words, given a circle, how can you find its center?

1.6 SKETCHPAD AND LOCUS PROBLEMS. The process of finding a set of points or its equation from a geometric characterization is called a locus problem. The 'Trace' and 'Locus' features of Sketchpad are particularly well adapted for this. The Greeks identified and studied the three types of conics: ellipses, parabolas, and hyperbolas. They are called conics because they each can be obtained by intersecting a cone with a plane. Here we shall use easier characterizations based on distance.

1.6.1 Demonstration: Determine the locus of a point P which moves so that

$$\text{dist}(P, A) = \text{dist}(P, B)$$

where A and B are fixed points.

The answer, of course, is that the locus of P is the perpendicular bisector of \overline{AB} . This can be proved synthetically using properties of isosceles triangles, as well as algebraically. But Sketchpad can be used to *exhibit* the locus by exploiting the ‘trace’ feature as follows.

- Open a new sketch and construct points A and B near the center of your sketch. Near the top of your sketch construct a segment \overline{CD} whose length is at least one half the length of \overline{AB} (by eyeballing).
- Construct a circle with center A and radius of length \overline{CD} . Construct another circle with center B and radius of length \overline{CD} .
- Construct the points of intersection between the two circles. (As long as your segment \overline{CD} is long enough they will intersect). Label the points P and Q . Select both points and under the **Display** menu select Trace Points. You should see a \surd next to it.
- Now drag C about the screen and then release the mouse. Think of the point C as the driver. What is the locus of P and Q ?
- Click on the screen. Your traced points should disappear. If you want a more permanent locus, we will have to use ‘Locus’ under the **Construct** menu. However, to use the ‘locus’

feature our driver must be constructed to lie on a path. An example to be discussed shortly will illustrate this.

End of Demonstration 1.6.1.

Now let's use Sketchpad on a locus problem where the answer is not so well known or so clear. Consider the case when the distances from P are not equal but whose ratio remains constant. To be specific, consider the following problem.

1.6.2 Exercise: Determine the locus of a point P which moves so that

$$\text{dist}(P, A) = 2 \text{ dist}(P, B)$$

where A and B are fixed points. (How might one modify the previous construction to answer this question?) Then, give the completion to **Conjecture 1.6.3** below.

1.6.3 Conjecture. Given points A and B , the locus of a point P which moves so that

$$\text{dist}(P, A) = 2 \text{ dist}(P, B)$$

is a/an _____.

A natural question to address at this point is: *How might one prove this conjecture?* More generally, what do we mean by a proof or what sort of proof suffices? Does it have to be a 'synthetic' proof, *i.e.* a two-column proof? What about a proof using algebra? Is a visual proof good enough? In what sense does Sketchpad provide a proof? A synthetic proof will be given in Chapter 2 once some results on similar triangles have been established, while providing an algebraic proof is part of a later exercise.

It is also natural to ask: *is there is something special about the ratio of the distances being equal to 2?*

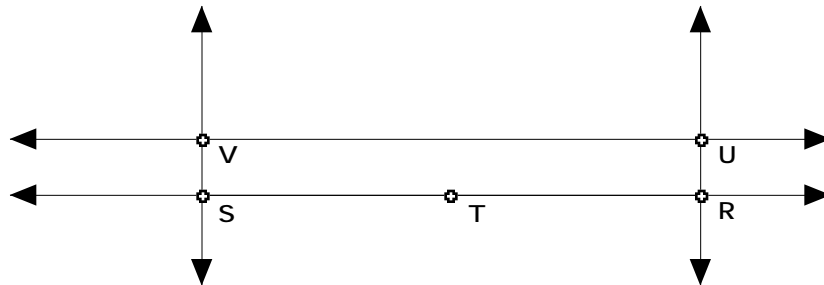
1.6.4 Exercise: Use Sketchpad to determine the locus of a point P which moves so that

$$\text{dist}(P, A) = m \text{ dist}(P, B)$$

where A and B are fixed points and $m = 3, 4, 5, \dots, 1/2, 1/3, \dots$. Use your answer to conjecture what will happen when m is an arbitrary positive number, $m \neq 1$? What's the effect of requiring $m > 1$? What happens when $m < 1$? How does the result of **Demonstration 1.6.1** fit into this conjecture?

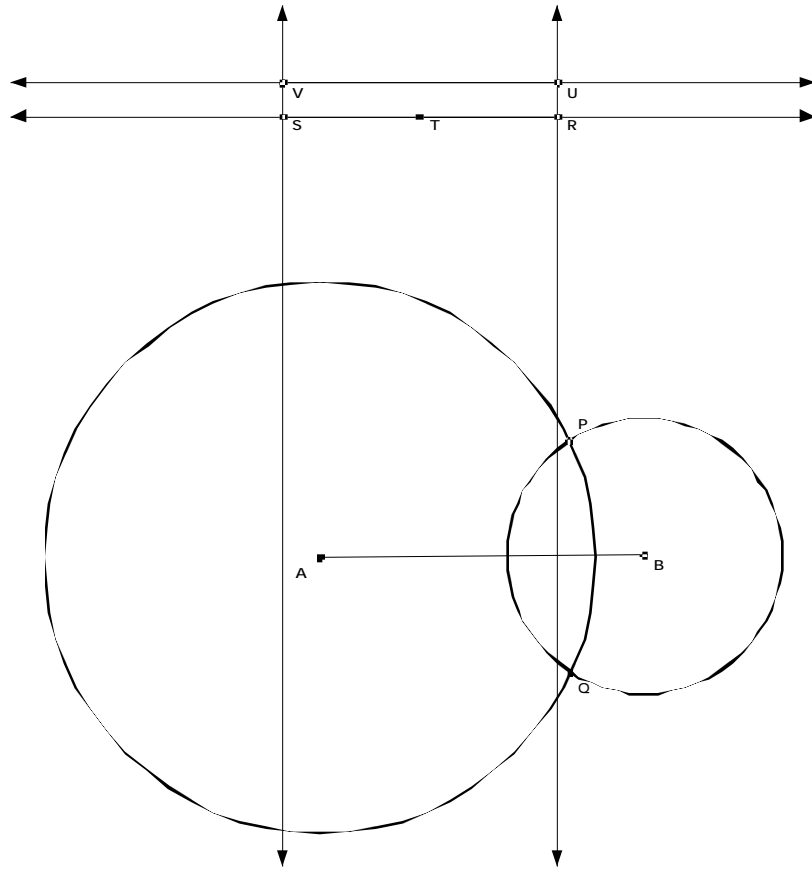
1.6.5 Demonstration, A Locus Example: In this Demonstration, we give an alternate way to examine Exercise 1.6.2 through the use of the Locus Construction. Note: to use “Locus” our driver point must be constructed upon a track. Open a new sketch and make sure that the Segment tool is set at Line (arrows in both directions).

- Draw a line near the top of the screen using the Line tool. Hide any points that are drawn automatically on this line. Construct two points on this line using the Point tool by clicking on the line in two different positions. Using the Text tool, label and re-label these two points as V and U (with V to the left of U). Construct the lines through U and V perpendicular to \overline{UV} . Construct a point on the perpendicular line through U . Label it R .
- Construct a line through R parallel to the first line you drew. Construct the point of intersection of this line with the vertical line through V using “Point At Intersection” from under the **Construct** menu. Label this point S . Construct the midpoint \overline{RS} . Label this point T . A figure similar to the following figure should appear on near the top of the screen.



This figure will be used to specify radii of circles. Also, the “driver point” will be U and the track it moves along is the line containing \overline{UV} .

- Towards the middle of the screen, construct \overline{AB} using the Segment tool. Construct the circle with center A and radius UV using “Circle By Center+Radius” from under the **Construct** menu. Construct the circle with center B and radius RT using “Circle By Center+Radius” from under the **Construct** menu. Construct both points of intersection of these two circles. Label or re-label these points P and Q . Both points have the property that the distance from P and Q to A is twice the distance from P and Q to B because the length of \overline{UV} is twice that of the length of \overline{RT} . The figure on screen should be similar to:



- Hide everything except \overline{AB} , the points of intersection P and Q of the two circles and the point U .
- Now select just the points P and U in that order. Go to “Locus” in the **Construct** menu. Release the mouse. What do you get? Repeat this construction with Q instead of P .

The “Locus” function causes the point U to move along the object it is on (here, line RS) and the resulting path of point P (and Q , in the second instance) is traced.

End of Demonstration 1.6.5.

Similar ideas can be used to construct conic sections. First recall their definitions in terms of distances:

1.6.6 Definition.

(a) An ellipse is the locus of a point P which moves so that

$$\text{dist}(P, A) + \text{dist}(P, B) = \text{const}$$

where A, B are two fixed points called the *foci* of the ellipse. Note: The word “*foci*” is the plural form of the word “*focus*.”

(b) A hyperbola is the locus a point P which moves so that

$$\text{dist}(P, A) - \text{dist}(P, B) = \text{const}$$

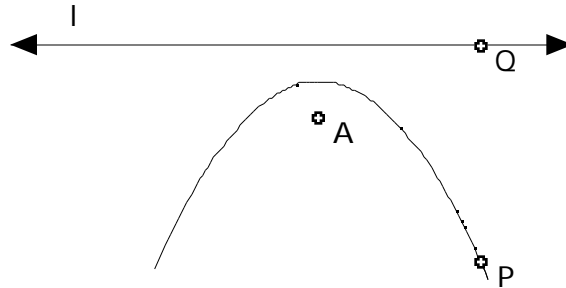
where A, B are two fixed points (the *foci* of the hyperbola).

(c) A parabola is the locus a point P which moves so that

$$\text{dist}(P, A) = \text{dist}(P, l)$$

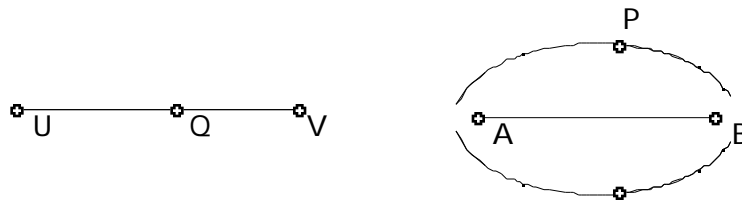
where A is a fixed point (the *focus*) and l is a fixed line (the *directrix*). Note: By $\text{dist}(P, l)$ we mean $\text{dist}(P, Q)$ where Q is on the line l and $\overline{P, Q}$ is perpendicular to l .

The points A and B are called the *foci* and the line l is called the *directrix*. The following figure illustrates the case of the parabola.



1.6.6a Demonstration: Construct an ellipse given points A, B for foci.

- c) Open a new sketch and construct points A, B . Near the top of your sketch construct a line segment \overline{UV} of length greater than AB . Construct a random point Q on \overline{UV} .
- Construct a circle with center at A and radius \overline{UQ} . Construct also a circle with center at B and radius \overline{VQ} . Label one of the points of intersection of these two circles by P . Thus $dist(P, A) + dist(P, B) = \overline{UV}$ (why?).
- Construct the other point of intersection the two circles. Now trace both points as you drag the point Q . Your figure should like



Why is the locus of P an ellipse?

The corresponding constructions of a hyperbola and a parabola appear in later exercises.

End of Demonstration 1.6.6a.

1.7 MORE SKETCHPAD AND CLASSICAL TRIANGLE GEOMETRY. We will continue to explore geometric ideas as we exploit the “script” feature of Sketchpad while looking at a sampling of geometry results from the 18th and 19th centuries. In fact, it's worth noting that many of the interesting properties of triangles were not discovered until the 18th, 19th, and 20th centuries despite the impression people have that geometry began and ended with the Greeks! Scripts will allow us to easily explore these geometric ideas by giving us the ability to repeat constructions without having to explicitly repeat each step.

1.7.1 Question: Given ABC construct the circumcenter, the centroid, the orthocenter, and the incenter. What special relationship do three of these four points share?

To explore this question via Sketchpad we need to start with a triangle and construct the required points. As we know how to construct the circumcenter and the other triangle points it would be nice if we did not have to repeat all of the steps again. Luckily, Scripts will provide the capability to repeat all of the steps quickly and easily. Now we will make a slight detour to learn about scripts then we will return to our problem.

There are two ways to construct a new script.

- Choose “New Script” from the **File** menu and record your construction as you go.
- Select all the objects in a construction for which you want to make a script. Then choose “Make Script” from the **Work** menu.

1.7.2 Script Demonstration: Create a script that will construct a Square-By-Edge.

Method 1: Recording as you go.

- Choose “New Script” from the **File** menu. A new window should open. Press the record button.
- Perform your construction for a Square-By-Edge. You should see your steps and the given objects being recorded in the script window. Once you are finished, hiding and labeling objects, if desired, press the stop button.

Method 2: Using Make Script on a previously constructed object.

- Start with a sketch that contains the desired construction, in this case a Square-By-Edge.
- Use the Arrow Tool to select all the objects for which you want to make a script. Remember you can click and drag using the Arrow Tool to select more than one object at once.

- Choose “Make Script” from the **Work** menu. You should see a script window open containing the (interpreted) steps and given objects for your construction.

It is always a good idea to add comments to your script, which describe the construction and the relationship between the givens and the constructed object. You can add comments to your script by moving the mouse over the comment bar (in between the buttons and the script steps) until you see a double headed arrow and then click and drag the arrow downward. Alternatively, you can select “Show Comments” from the **Edit** menu. You can save your script as you would any file, just make sure it is the active window when you select Save File. (Scripts should end with the file extension .gss.)

To play your script you can do the following.

- Open the script and a sketch.
- Create and select the objects that match the Givens in the script in the order they are listed.
- Click on playback button in the script window.

Play - plays back the script step by step with speed as set in the Display menu.

Fast - plays back the script so quickly you cannot follow the steps in the construction.

Step - plays back the script step-by-step as you keep pressing the Step button.

You can also access your scripts through the toolbar. This is especially useful when you are using many scripts and find it tedious to keep track of all the windows. In order to make the Script Tool appear on your toolbar you need to direct Sketchpad to the appropriate directory. For this reason, it is probably best to save all of your scripts to the same directory. Sketchpad comes with a set of useful Sample Scripts that we will load now.

- Under the **Display** menu select “Preferences” and then click on More...
- Under the Script Tool Folder heading click on “Set”. (If the Script Tool Folder had been previously set you would need to “Clear” it first.)
- Find the Directory entitled “Sample Scripts” which comes with Geometer’s Sketchpad and click on the “Select Sample Scripts” button.
- Click on Continue and then Okay.

You should now see the Script Tool icon in your toolbox. Press the button to see the tools available. The first line of the comment box is displayed as the name. Select a script to use it and you can select the givens one at a time by pointing to existing or new objects in your sketch.

End of Script Demonstration 1.7.2.

1.7.2a Exercise: Write scripts to construct each of the following, given the vertices A , B , and C of ABC :

- a) the circumcenter of ABC
- b) the centroid of ABC
- c) the orthocenter of ABC
- d) the incenter of ABC .

1.7.2b Exercise: In a new sketch draw triangle ABC . Construct the circumcenter of ABC and label it O . Construct the centroid of ABC and label it G . Construct the orthocenter of ABC and label it H . What do you notice? Confirm your observation by dragging each of the vertices A , B , and C . Complete **Conjectures 1.7.3** and **1.7.4** and also answer the questions posed in the text between them.

1.7.3 Conjecture. (Attributed to Leonhard Euler in 1765) For any ABC the circumcenter, orthocenter, and centroid are

_____.

Hopefully you will not be satisfied to stop there! Conjecture 1.7.3 suggested O , G , and H are collinear, that is they lie on the so-called **Euler Line** of a triangle. What else do you notice about O , G , and H ? Don't forget about your ability to measure lengths and other quantities. What happens when ABC becomes obtuse? When will the Euler line pass through a vertex of ABC ?

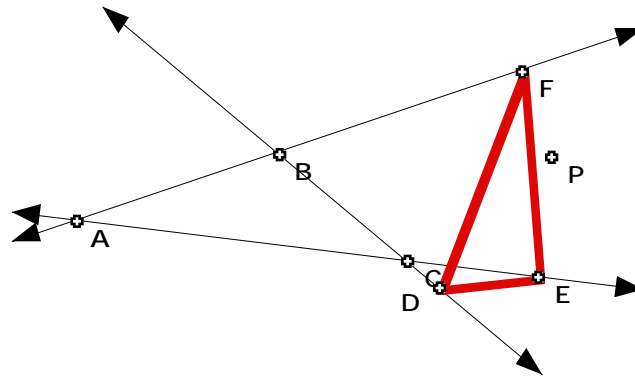
1.7.4 Conjecture. The centroid of a triangle bisects / trisects (Circle one) the segment joining the circumcenter and the orthocenter.

End of Exercise 1.7.2b.

Another classical theorem in geometry is the so-called Simson Line of a triangle, named after the English mathematician Robert Simson (1687-1768). The following illustrates well how Sketchpad can be used to discover such results instead of being given them as accepted facts. We begin by exploring Pedal triangles.

1.7.5 Demonstration on the Pedal Triangle:

- In a new sketch construct three non-collinear points labeled A , B , and C and then construct the three lines containing segments \overline{AB} , \overline{BC} , and \overline{AC} . (We want to construct a triangle but with the sides extended.) Construct a point P anywhere in your sketch.
- Construct the perpendicular from P to the line containing \overline{BC} and label the foot of the perpendicular as D . Construct the perpendicular from P to the line containing \overline{AC} then and the foot of the perpendicular as E . Construct the perpendicular from P to the line containing \overline{AB} and label the foot of the perpendicular as F .
- Construct $\triangle DEF$. Change the color of the sides to red. $\triangle DEF$ is called the pedal triangle of $\triangle ABC$ with respect to the point P .



End of Demonstration 1.7.5.

1.7.5a Exercise: Make a Script which constructs the Pedal Triangle $\triangle DEF$ for a given point P and the triangle with three given vertices A , B and C . (Essentially, save the script constructed in **Demonstration 1.7.5** as follows: Hide everything except the pedal triangle $\triangle DEF$ and the point P . Select those objects with the Selection tool. Then choose “Make Script” from the **Work** menu. Note that in running this script, the points A , B , and C must be selected first and then point P must be selected.)

Now you can start exploring with your pedal triangle script.

1.7.5b Exercise: When is $\triangle DEF$ similar to $\triangle ABC$? Can you find a position for P for which $\triangle DEF$ is equilateral? Construct the circumcircle of $\triangle ABC$ and place P close to or even on the circumcircle. Complete **Conjecture 1.7.6** below.

1.7.6 Conjecture. P lies on the circumcircle of $\triangle ABC$ if and only if the pedal triangle is

We shall turn to the proof of some of these results in Chapter 2.

1.8 Exercises. In these exercises we continue to work with Sketchpad, including the use of scripts. We will look at some problems introduced in the last few sections as well as discover some new results. Later on we'll see how Yaglom's Theorem and Napoleon's Theorem both relate to the subject of tilings.

Exercise 1.8.1, Some algebra: Write down the formula for the distance between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the coordinate plane. Now use coordinate geometry to prove the assertion in Conjecture 1.6.3 (regarding the locus of P when $\text{dist}(P, A) = 2 \text{dist}(P, B)$) that the locus is a circle. To keep the algebra as simple as possible assume that $A = (-a, 0)$ and $B = (a, 0)$ where a is fixed. Set $P = (x, y)$ and compute $\text{dist}(P, A)$ and $\text{dist}(P, B)$. Then use the condition $\text{dist}(P, A) = 2 \text{dist}(P, B)$ to derive a relation between x and y . This relation should verify that the locus of P is the conjectured figure.

Exercise 1.8.2, Locus Problems.

(a) Given points A, B in the plane, use Sketchpad to construct the locus of the point P which moves so that

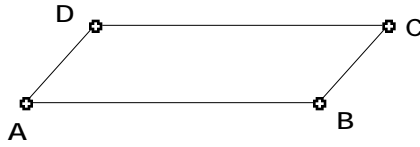
$$\text{dist}(P, A) - \text{dist}(P, B) = \text{constant}.$$

(b) Given point A and line l in the plane, use Sketchpad to construct the locus of the point P which moves so that

$$\text{dist}(P, A) = \text{dist}(P, l).$$

Hint: Construct a random point Q on the line l . Then think about relationship between Q and A to P and use that to find the corresponding point P on the parabola.

Exercise 1.8.3, Yaglom's Theorem. In a new sketch construct any parallelogram $ABCD$.



- On side \overline{AB} construct the outward pointing square having \overline{AB} as one of its sides. Construct the center of this square and label it Z .
- Construct corresponding squares on the other sides \overline{BC} , \overline{CD} , and \overline{DA} , and label their centers X , U , and V respectively.

What do you notice? Confirm your observation(s) by dragging the vertices of the original parallelogram.

Exercise 1.8.4, Miquel Point. In a new sketch draw an acute triangle ABC .

- On side \overline{AB} select a point and label it D . Construct a point E on side \overline{BC} , and a point F on side \overline{CA} .
- Construct the circumcircles of ADF , BDE , and CEF .

What do you notice? Confirm your observation(s) by dragging each of the points A , B , C , D , E , and F . Now drag vertex A so that ABC becomes obtuse. Do your observation(s) remain valid or do they change for obtuse triangles? What happens if the three points D , E , and F are collinear (allow D , E , and F to be on the extended sides of the triangle)?

Exercise 1.8.5, Napoleon's Theorem. In a new sketch draw any acute triangle ABC .

- On side \overline{AB} construct the outward pointing equilateral triangle having \overline{AB} as one of its sides. Construct the corresponding equilateral triangle on each of \overline{BC} , and \overline{CA} .
- Construct the circumcircle of each of the equilateral triangles just constructed.

What do you notice? Confirm your observation(s) by dragging the vertices of ABC .

Exercise 1.8.6. Open a new sketch and construct an equilateral triangle ABC . Select any point P on the triangle or in its interior.

- Construct the perpendicular segment from P to each of the sides of ABC .

- Measure the length of the segment from P to \overline{BC} ; call it r_a . Measure the length of the segment from P to \overline{CA} ; call it by r_b . Measure the length of the segment from P to \overline{AB} ; call it r_c . Compute the sum $r_a+r_b+r_c$.

Drag P around to see how the value of $r_a+r_b+r_c$ changes as P varies. What do you notice?

Explain your answer by relating the value you have obtained to some property of $\triangle ABC$. (Hint: look first at some special locations for P .)

Exercise 1.8.7. Confirm your observation in Conjecture 1.7.3 regarding the Euler Line for the special case of the triangle $\triangle ABC$ in which $A = (-2, -1)$, $B = (2, -1)$, and $C = (1, 2)$. That is, find the coordinates of the circumcenter O , the centroid G and the orthocenter H using coordinate geometry and show that they all lie on a particular straight line.

1.9 SKETCHPAD AND COORDINATE GEOMETRY. Somewhat surprisingly perhaps, use of coordinate geometry and some algebra is possible with Sketchpad. For instance, graphs defined by $y = f(x)$ or parametrically by $(x(t), y(t))$ can be drawn by regarding the respective variable x or t as a parameter on a fixed curve. Graphs can even be drawn in polar coordinates.

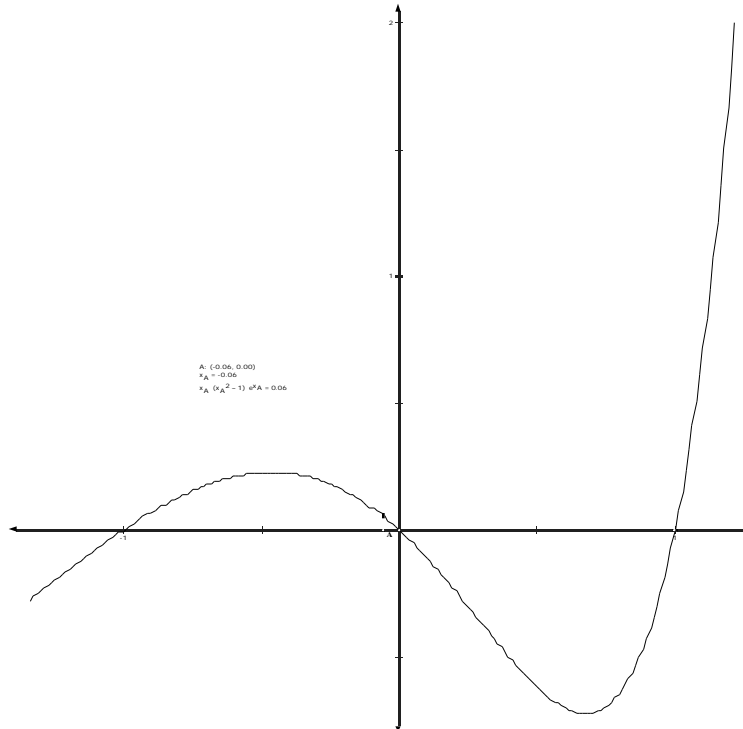
1.9.1 Demonstration: As an illustration let's consider the problem of drawing the graph of $y = 2x + 1$; it is a straight line having slope 2 and y-intercept 1 and the points on the graph have the form $(x, 2x + 1)$. Sketchpad draws this graph by constructing the locus of $(x, 2x + 1)$ as x varies over a portion of the x -axis. This can be done via the **Trace Point** or **Locus** feature described earlier, but it can also be done using the **Animation** feature as follows.

- Open a new sketch and from the **Graph** menu and choose "Create Axes". Select the x -axis and construct a point on this axis using the "Construct Point on Object" tool from the **Construct** menu. Label this point A .
- To graph the function we want to let A vary along the x -axis so let's illustrate the animation feature first. Select A and the x -axis. From the **Edit** menu choose "Action" and then move the cursor over to the right and select "Animation". A menu called Path Match will appear - by default the menu will read "Point A moves bidirectionally along the x -axis quickly". Select "Animate". An animation button will appear in the sketch. Double click on it to start or stop the animation. Try this.
- Select point A and then select "Coordinates" from the **Measure** menu. The coordinates of A will appear on the screen. Select these coordinates and then select the "Calculator" from the **Measure** menu. Drag down the 'Values' icon until you reach 'x'-coordinate. The x -coordinate of A will now appear on the screen.

- You are now ready to begin graphing. Open the “Calculator”. This is used to define whatever function is to be graphed, say the function $2x + 1$. Type in the box on the calculator whatever function of x you want to graph, clicking on the x_A -coordinate on the screen for the x -variable in your function.
- To plot a point on the graph of $y = 2x + 1$ select x_A and $2x_A + 1$ from the screen and then select “Plot as (x, y) ” from the **Graph** menu. This plots a point on the coordinate plane on your screen. Select this point and then select “Trace Point” from the **Display** menu. If you want, you can color the point so that the graph will be colored when you run the animation! Now double click on the “Animate” button on screen and watch the graph evolve.
- Notice that Cartesian or polar coordinates can be chosen. For Cartesian coordinates we used the x -axis as driver, but any other line would do; for polar coordinates the natural curve to choose is a circle centered at the origin. For simplicity we stuck to the Cartesian case of a point on the x -axis.

End of Demonstration 1.9.1.

1.9.1a Exercise. Use the construction detailed above to draw the graph of $y = x(x^2 - 1)e^x$ as shown below.



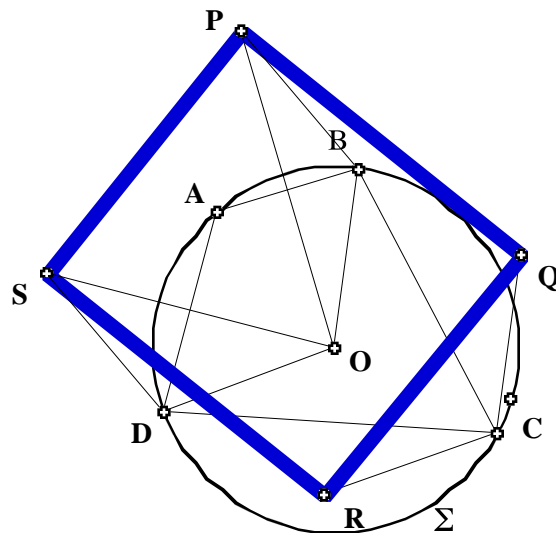
Further examples are given in the later exercises.

1.10 AN INVESTIGATION VIA SKETCHPAD. As a final illustration of the possibilities for using Sketchpad before we actually begin the study of various geometries, let us see how it might be used in problem-solving and then be supplemented by more traditional coordinate geometry and synthetic methods.

1.10.1 Demonstration. Let A, B, C and D be four distinct points on a circle whose center is O . Now let P, Q, R and S be the mirror images of O in the respective chords $\overline{AB}, \overline{BC}, \overline{CD}$ and \overline{DA} of . Investigate the properties of the quadrilateral $PQRS$. Justify algebraically or synthetically any conjecture you make. Investigate the properties of the corresponding triangle PQR when there are only three distinct points on .

One natural first step in any problem-solving situation is to draw a picture if at all possible - in other words to realize the problem as a visual one.

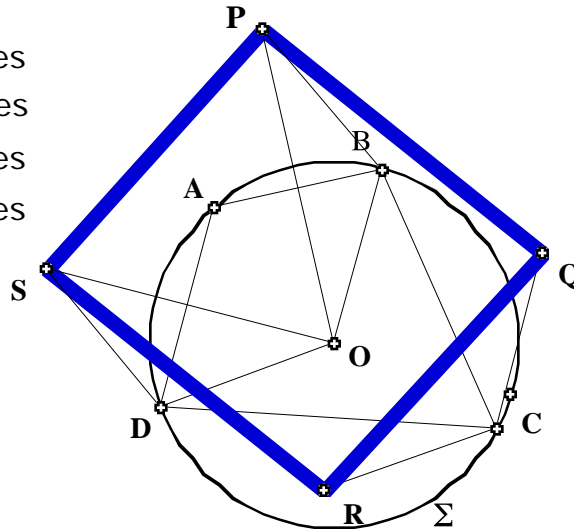
- Open a new sketch. Draw a circle and label it Σ . Note the point on the circle which when dragged allows the radius of the circle to be varied - this will be useful in checking if a conjecture is independent of a particular Σ .
- Construct four random points on Σ and label them A, B, C and D . Construct the corresponding chords $\overline{AB}, \overline{BC}, \overline{CD}$ and \overline{DA} of Σ . Make sure that A, B, C and D can be moved freely and independently of each other - this will be important in testing if a conjecture is independent of the location of A, B, C and D .
- To construct the mirror image P of O in \overline{AB} select \overline{AB} and then double click on it - the small squares denoting that \overline{AB} has been selected should 'star-burst'. Alternatively, drag down on the **Transform** menu and select "mark mirror" indicating that reflections can be made with respect to \overline{AB} .
- Select O and drag down on the **Transform** menu until "Reflect" is highlighted. The mirror image of O in the mirror \overline{AB} will be constructed. Label it P . Repeat this to construct the respective mirror images Q, R and S . Draw line segments to complete the construction of the quadrilateral $PQRS$. Your figure should look similar to the following



The problem is to decide what properties quadrilateral $PQRS$ has. Sketchpad is a particularly good tool for investigating various possibilities. For example, as drawn, it looks as if its side-lengths are all equal. To check this, measure the lengths of all four sides of $PQRS$. Immediately we see that adjacent sides do not have the same length, but opposite sides do. Drag each of A, B, C and D as well as the point specifying the radius of Σ to check if the equality $PQ=RS$ does not depend on the location of these points or the radius of Σ . In the figure as

drawn the side \overline{SP} looks to be parallel to the opposite side \overline{RQ} . To check this measure angles $\angle RSP$ and $\angle SRQ$. Your figure should now look like:

$m \overline{QR} = 1.69$ inches
 $m \overline{SP} = 1.69$ inches
 $m \overline{PQ} = 1.87$ inches
 $m \overline{SR} = 1.87$ inches
 $m\angle RSP = 86^\circ$
 $m\angle SRQ = 94^\circ$



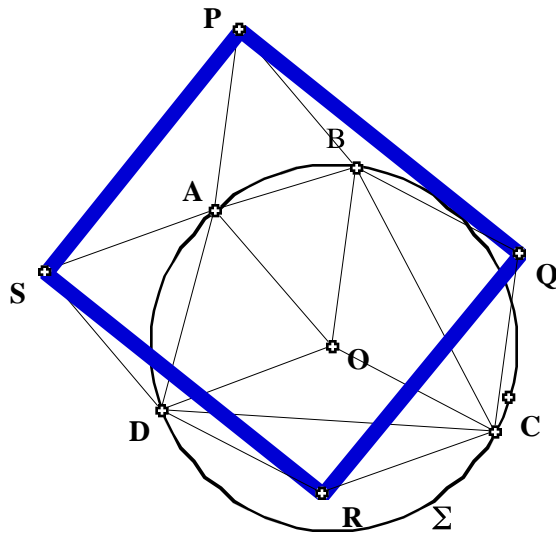
Since $m \angle RSP + m \angle SRQ = 180^\circ$, this provides evidence that \overline{SP} is parallel to \overline{RQ} though it does not prove it of course (why?). To check if the sum is always 180° drag each of A, B, C and D as well as the point specifying the radius of Σ .

End of Demonstration 1.10.1.

All this Sketchpad activity thus suggests the following result.

1.10.2 Theorem. Let A, B, C and D be four distinct points on a circle whose center is O . Then the mirror images P, Q, R and S of O in the respective chords $\overline{AB}, \overline{BC}, \overline{CD}$ and \overline{DA} of are always the vertices of a parallelogram $PQRS$.

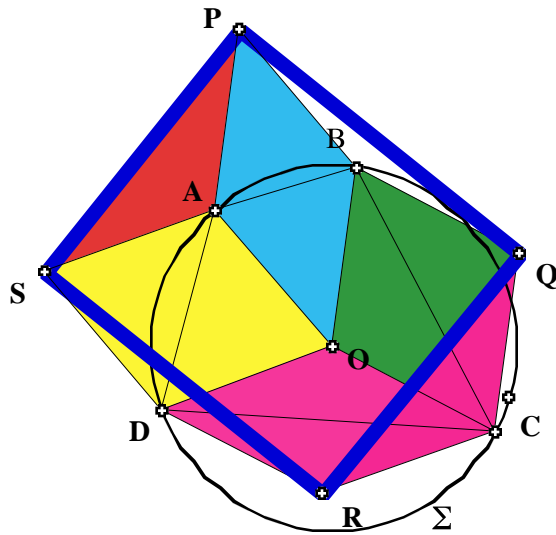
While Sketchpad has provided very strong visual support for the truth of Theorem 1.10.2, it hasn't supplied a complete proof (why?). For that we have to use synthetic methods or coordinate geometry. Nonetheless, preliminary use of Sketchpad often indicates the path that a formal proof may follow. For instance, in the figure below



a number of line segments have been added. In particular, the lengths of the line segments \overline{OA} , \overline{OB} , \overline{OC} and \overline{OD} are equal because each is a radial line of Σ . This plus visual inspection suggests that each of

$$OAPB, OBQC, OCRD, ODSA$$

is a rhombus and that they all have the same side length, namely the radius of Σ . Assuming that this is true, how might it be used to prove Theorem 1.10.2? Observe first that to prove that $PQRS$ is a parallelogram it is enough to show that $PS=QR$ and $PQ=SR$ (why?). To prove that $PS=QR$ it is enough to show that $\triangle SAP \cong \triangle RCQ$. This is a good point to draw a clear diagram



illustrating what has been discussed because a good diagram isolating the key features of a figure often helps with a proof. One may color the interior of each rhombus for example, by selecting the vertices in order then by choosing “Polygon Interior” from the **Construct** menu. You may construct the interior of any polygon in this manner. You can change the color of the interior by selecting the interior and then by using the **Display** menu. Indeed, to show that

$\triangle SAP \cong \triangle RCQ$ we can use (SAS) because

$$SA=RC, \quad AP=CQ$$

since all four lengths are equal to the radius of Σ , while

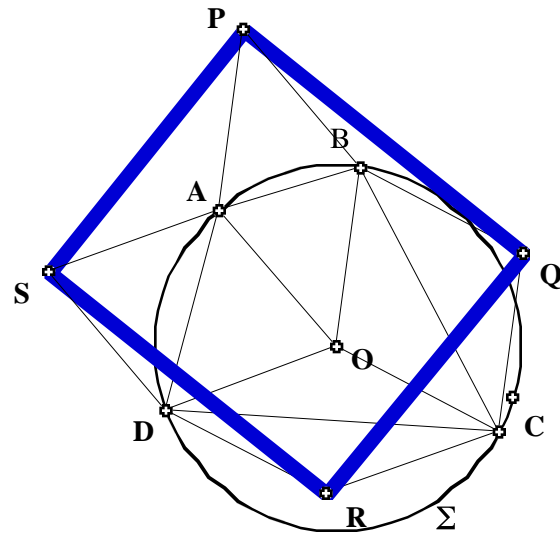
$$m \angle SAP = m \angle DOB = m \angle RCQ.$$

Consequently, the key property needed in proving Theorem 1.10.2 is the fact that each of

$$\triangle OAPB, \triangle OBQC, \triangle OCRD, \triangle ODSA$$

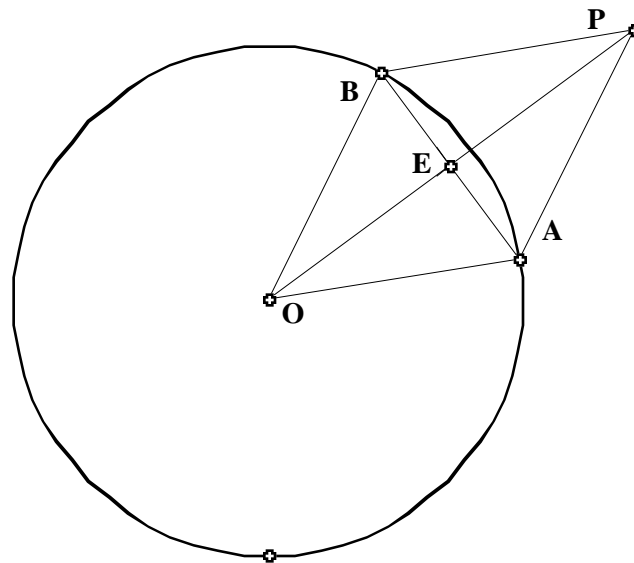
is a rhombus. Although this still doesn't constitute a complete proof (why?), it does illustrate how drawing accurate figures with Sketchpad can help greatly in visualizing the steps needed in a proof.

Proof of Theorem 1.10.2. (Synthetic) Recall the earlier figure



To prove that $PQRS$ is a parallelogram it is enough to show that $PS=QR$ and $PQ=SR$. We prove first that $PS=QR$ by showing that $\angle SAP = \angle RCQ$.

Step 1. The construction of P ensures that $OAPB$ is a rhombus. Indeed, in the figure



$OA=OB$, $\angle AEO = 90^\circ$ and $OE=EP$. Thus, by (HL),

$$\angle AEO = \angle BEO = \angle AEP = \angle BEP .$$

Consequently, $OA=OB = AP=AQ$ and so $OAPB$ is a rhombus. Similarly, each of $OBQC$, $OCRD$ and $ODSA$ is rhombus; in addition, they all have the same side length since $OA=OB=OC=OD$.

Step 2. Now consider SAP and RCQ . By step 1, $SA=AP$ and $RC=CQ$. On the other hand, because Step 1 ensures that \overline{SA} is parallel to both \overline{DO} and \overline{RC} , while \overline{PA} is parallel to \overline{BO} and \overline{QC} , it follows that

$$m \angle SAP = m \angle DOB = m \angle RCQ.$$

Hence $SAP \cong RCQ$ by (SAS).

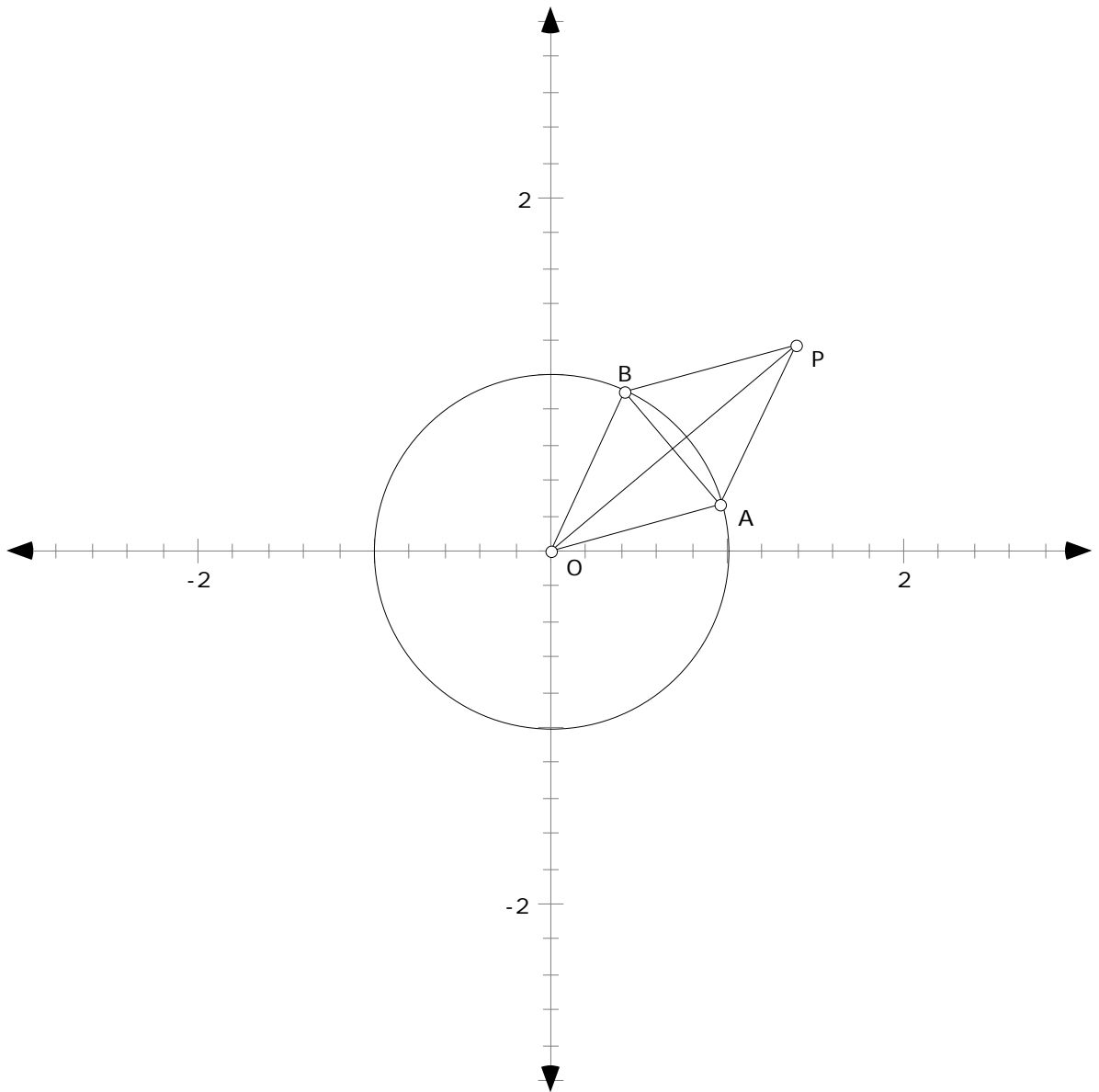
Step 3. In exactly the same way as in Step 2, by showing that $PBQ \cong SCR$, we see that $PQ=SR$. Hence $PQRS$ is a parallelogram, completing the proof of Theorem 1.10.2. **QED**

As often happens, a coordinate geometry proof of Theorem 1.10.2 is shorter than a synthetic one - that's one reason why it's often a smart idea to try first using algebra when proving a given result! Nonetheless, an algebraic proof often calls for good algebra skills!

Proof of Theorem 1.10.2. (Algebraic) Now the idea is to set up the algebra in as simple a form as possible. So take the unit circle $x^2 + y^2 = 1$ for \mathcal{C} ; the center O of \mathcal{C} is then the origin. As any point on \mathcal{C} has the form $(\cos\theta, \sin\theta)$ for a choice of θ with $0 \leq \theta < 2\pi$, we can assume that

$$A = (\cos\theta_1, \sin\theta_1), \quad B = (\cos\theta_2, \sin\theta_2) \quad C = (\cos\theta_3, \sin\theta_3) \quad D = (\cos\theta_4, \sin\theta_4),$$

and if we assume that $0 \leq \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi$, then points on \mathcal{C} will be in counter-clockwise order. The first thing to do now is calculate the coordinates of the mirror images of the origin in the respective chords $\overline{AB}, \overline{BC}, \overline{CD}$ and \overline{DA} of \mathcal{C} ; again a picture helps:



Then

$$E = \left(\frac{1}{2} \cos \theta_1 + \frac{1}{2} \cos \theta_2, \frac{1}{2} \sin \theta_1 + \frac{1}{2} \sin \theta_2 \right),$$

since E is the midpoint of \overline{AB} , while

$$P = (\cos \theta_1 + \cos \theta_2, \sin \theta_1 + \sin \theta_2),$$

since $OE=EP$. (Notice that here again we are really using the fact $OAPB$ is a rhombus.)
Similarly,

$$Q = (\cos\theta_2 + \cos\theta_3, \sin\theta_2 + \sin\theta_3), \quad R = (\cos\theta_3 + \cos\theta_4, \sin\theta_3 + \sin\theta_4)$$

and

$$S = (\cos\theta_4 + \cos\theta_1, \sin\theta_4 + \sin\theta_1).$$

To show that $PQRS$ is a parallelogram it is enough to show that \overline{PQ} and \overline{SR} have equal slope, and that \overline{QR} and \overline{PS} have equal slope. But

$$\text{slope}\overline{PQ} = \frac{\sin\theta_3 - \sin\theta_1}{\cos\theta_3 - \cos\theta_1} = \text{slope}\overline{SR}.$$

Similarly,

$$\text{slope}\overline{QR} = \frac{\sin\theta_4 - \sin\theta_2}{\cos\theta_4 - \cos\theta_2} = \text{slope}\overline{PS}.$$

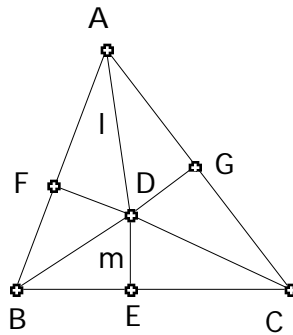
Hence $PQRS$ is a parallelogram completing an algebraic proof of Theorem 1.10.2. **QED**

The algebraic proof is undoubtedly simpler than the synthetic proof, but neither gives an explanation of *why* the result is true *which is what a good proof should do*. For this we shall have to wait until Chapter 2. Other investigative problems of a similar nature are given in the problems for this chapter.

1.11 FALSE THEOREMS We all know the power a figure can provide when trying to understand why certain geometry results are true. In fact, they seem to be essential for understanding and providing proofs. One of the most crucial features of dynamic geometry software is the possibility it provides for ‘dragging’ a given construction to provide many different views of the same setting. This can be used for investigation of a given problem with a view to formulating a conjecture or it can be used in the proof of a given conjecture. There is, however, a danger in relying on sketches, extra assumptions may be added by relying on the sketch, special cases may be omitted, or absurd results can be derived from an inaccurate sketch. We’ll look at two examples of where figures can be deceiving.

1.11.1 False Theorem: All triangles are isosceles.

Proof: Let ABC be a triangle with l the angle bisector of $\angle A$, m the perpendicular bisector of BC cutting BC at midpoint E , and D the intersection of l and m . From D , draw perpendiculars to AB and AC , cutting them at F and G , respectively. Finally, draw DB and DC . The following figure shows a sketch of the situation.

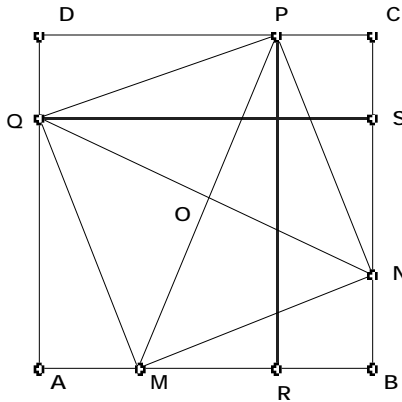


$\triangle ADF \cong \triangle ADG$ (AAS), so $AF = AG$ and $DF = DG$. $\triangle BDE \cong \triangle CDE$ (SAS), so $BD = CD$. This implies $\triangle BDF \cong \triangle CDG$ (HL), so $FB = GC$. Thus $AB = AF + FB = AG + GC = AC$, and so $\triangle ABC$ is isosceles. **QED**

1.11.2 Exercise: What is wrong with the proof of Theorem 1.11.1? Absolutely nothing is wrong with the chain of reasoning, so where does the problem lie? Try constructing the given configuration on Sketchpad. What do you observe?

1.11.3 False Theorem: Any rectangle inscribed in a square must itself be a square.

Proof. Consider the following picture of a rectangle $MNPQ$ inscribed in a square $ABCD$



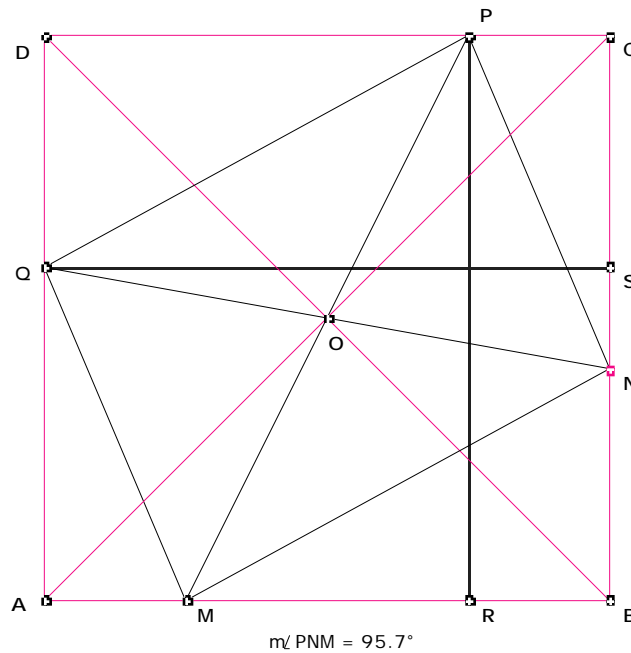
The rectangle $PQMN$ certainly looks like a square doesn't it? To prove that it is we'll show that the diagonals of $PQMN$ are perpendicular since the only rectangles having perpendicular diagonals are squares. Construct the point of intersection of the diagonals of rectangle $MNPQ$; label it O . Construct the perpendicular \overline{PR} to \overline{AB} ; construct also the perpendicular \overline{QS} to \overline{BC} . Then $\overline{PR} \parallel \overline{QS}$. Since the diagonals of any rectangle are congruent, $\overline{PM} \cong \overline{QN}$. So $\triangle PMR \cong \triangle QNS$, and hence $\angle PMR \cong \angle QNS$. Now consider the quadrilateral $MBNO$. Its exterior angle at vertex N is congruent to the interior angle at vertex M , so the two interior angles at vertices N and M are supplementary. Thus the interior angles at vertices B and O must be supplementary. But $\angle ABC$ is a right angle and hence $\angle NOM$ must also be a right angle. Therefore the diagonals of the rectangle $MNPQ$ are perpendicular; ensuring that $MNPQ$ is a square.

QED

1.11.4 Exercise. What is wrong with the proof of Theorem 1.11.4? Nothing actually; every step is logically correct! We could use Sketchpad to explore the various possibilities for inscribing a rectangle in a square by using the 'dragging' feature.

1.11.5 Demonstration: Construct a figure to reveal the error in the proof of Theorem 1.11.1. Open a new sketch and draw a square $ABCD$; draw also the diagonals of this square and label

their point of intersection by O . To vary the inscribed rectangle $MNPQ$ of the proof by dragging we want to construct fixed opposite vertices, say P and Q , but construct opposite vertices M and N so that they vary as we drag N say. Select a point P on side \overline{CD} and draw the line passing through O and P ; label by M its point of intersection with \overline{AB} . Hide the line and then construct \overline{PM} . Now select a point N on \overline{BC} . The idea now is that everything constructed starting from N will move as the point N is dragged along \overline{BC} , while nothing that was constructed before will move. Draw the line through N and O to determine the remaining vertex Q on \overline{DA} ; hide the line through N and O . Construct \overline{NQ} ; this ensures that vertex Q will move as N moves. Finally, draw the line segments \overline{PN} , \overline{NM} , \overline{QP} , and \overline{QM} as well as the perpendiculars \overline{PR} and \overline{QS} . This gives the following figure



To check if the inscribed figure is a rectangle, measure the angle $\angle PNM$.

Drag the point N along \overline{BC} . Does the figure $MNPQ$ move? Investigate when $MNPQ$ is a square. When is it a rectangle, but not a square? When is it neither a square nor a rectangle?

Formulate a conjecture describing conditions under which $MNPQ$ is a rectangle. Prove your conjecture! Why are all the steps in the proof of the theorem above correct, yet the result is incorrect?

End of Demonstration 1.11.5.

The previous two examples should not suggest the banning of any figures but instead stress the need for accurately drawn figures, rather than quick sketches. By using Sketchpad we construct accurate figures and consider many different cases, thus reducing the likelihood of overlooking something. The first false theorem (all triangles are isosceles) could have been totally avoided if we had started with an accurately drawn figure. The second false theorem (any rectangle inscribed in a square is itself a square) can be avoided by the use of the dragging feature.

1.12 Exercises. The following provide multiple viewpoints on geometry: synthetic, algebraic and dynamic.

The various circles associated with a triangle may seem to involve interesting but non-practical ideas. This is not the case as the following problems show. Indeed, one of the points we shall be emphasizing throughout this course is that the study of geometry is important as a study in logical analysis, but it is also very important for the uses that can be made of geometry. We look first at a problem involving the incircle of a triangle.

Exercise 1.12.1. What is the largest sphere that will pass through a triangular hole whose sides are 7 in., 8 in., and 9 in. long?

The next problem involves the circumcircle of a triangle.

Exercise 1.12.2. A thin triangular-shaped iron plate is accidentally dropped into a hemispherical tank, which is 10 ft. deep and full of water. It is noticed that the iron triangle is lying parallel to the surface of the water, so it is proposed to retrieve the triangle by lowering a powerful magnet into the tank at the end of a rope. What is the minimum length of rope needed if the shortest side of the triangle is 10 ft. long and the angles of the triangle are 45, 60, and 75 degrees

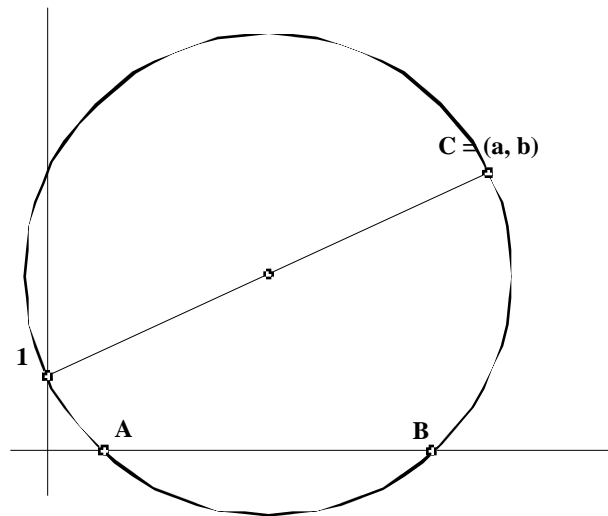
Exercise 1.12.3. Using Sketchpad, open a new sketch and draw a triangle ABC .

- On side \overline{BC} construct the outward pointing square having \overline{BC} as one of its sides; construct the center of this square and label it X .
- Construct the corresponding outward pointing squares on \overline{CA} and \overline{AB} ; label their respective centers Y and Z .
- Construct the segments \overline{AX} and \overline{YZ} .

What properties do \overline{AX} and \overline{YZ} have? Check your conjecture by dragging the vertices of ABC around.

Exercise 1.12.4. Quadratic equations: The Greeks used geometry where now we would use algebra. For instance, they knew how to construct the roots of the quadratic equation

$x^2 - ax + b = 0$ for given values of a, b when $a^2 > 4b$. Let's use Sketchpad to illustrate their method. Draw a pair of perpendicular lines, which we'll think of as the x - and y -axes. On the y -axis choose a fixed point and label it 1; the distance of this point from the point of intersection of the two perpendicular lines is to be thought of as specifying what unit length means. Given a, b draw the point C having (a, b) as coordinates as well as the point on the y -axis having y -coordinate 1. Now draw the circle having the line segment from this point on the y -axis to C as a diameter. Finally, label the points of intersection of this circle with the x -axis by A and B . You should have a figure looking like



Show that the x -coordinates of A, B are the roots of the equation $x^2 - ax + b = 0$. Where did the condition $a^2 > 4b$ get used? What is the equation of the circle you drew? This all looks pretty straightforward to us now that we have the analytic geometry of circles available to us, but it should be remembered that almost 2,000 years elapsed after the Elements were written before Descartes combined algebra with geometry!

Exercise 1.12.5. A gardener cut a piece of sod to fill a triangular hole in a grass lawn. When he came to put the grass sod in the hole he found that it fit perfectly, but only with the wrong side up. To fit the sod in the triangular hole with the right side up he had to cut it. How did he cut it into three pieces so that the shape of each piece was unchanged when he turned it over?

Exercise 1.12.6. Birthday Cake: For her birthday party, Sally's father baked a chocolate cake in the shape of a triangular prism. Sally will have eight of her friends at her birthday party, and

everyone likes chocolate cake and icing. How is Sally to cut the cake efficiently so that she and each of her friends get equal shares of cake and icing?

Exercise 1.12.7. Using Sketchpad, in a new sketch draw any convex quadrilateral $ABCD$.

Recall that a convex quadrilateral is one that has all interior angles less than 180° .

- On side \overline{AB} construct the outward pointing square having \overline{AB} as one of its sides. Construct the center of this square and label it Z .
- Construct corresponding squares on the other sides \overline{BC} , \overline{CD} , and \overline{DA} , and label their centers X , U and V respectively.
- Draw the line segments \overline{ZU} and \overline{XV} . Make a conjecture about the properties of \overline{ZU} and \overline{XV} . Check these properties by dragging the vertices of the quadrilateral $ABCD$. Drag one of the vertices so that the quadrilateral becomes concave. Do the properties of \overline{ZU} and \overline{XV} still hold true or do they change for concave quadrilaterals?