<center>**Chapter 4**</center>

TRANSFORMATIONS

4.1 TRANSFORMATIONS, ISOMETRIES. The term *transformation* has several meanings in mathematics. It may mean any change in an equation or expression to simplify an operation such as computing a derivative or an integral. Another meaning expresses a functional relationship because the notion of a function is often introduced in terms of a mapping

$$f : A \quad B$$

between sets A and B; for instance, the function $y = x^2$ can be thought of as a mapping $f : x \quad x^2$ of one number line into another. On the other hand, in linear algebra courses a *linear* transformation maps vectors to vectors and subspaces to subspaces. When we use the term transformation in geometry, however, we have all of these interpretations in mind, plus another one, namely the idea that the transformation should map a geometry to a geometry. A formal definition makes this precise.

Recall first that if $f : A \quad B$ is a mapping such that every point in the range of f has a unique pre-image in A, then f is said to be *one to one* or *injective*. If the range of f is all of B, then f is said to be *onto* or *surjective*. When the function is both one to one and onto, it is called a *bijection* or is said to be *bijective*. The figures below illustrate these notions pictorially.

<center>one to one   onto</center>
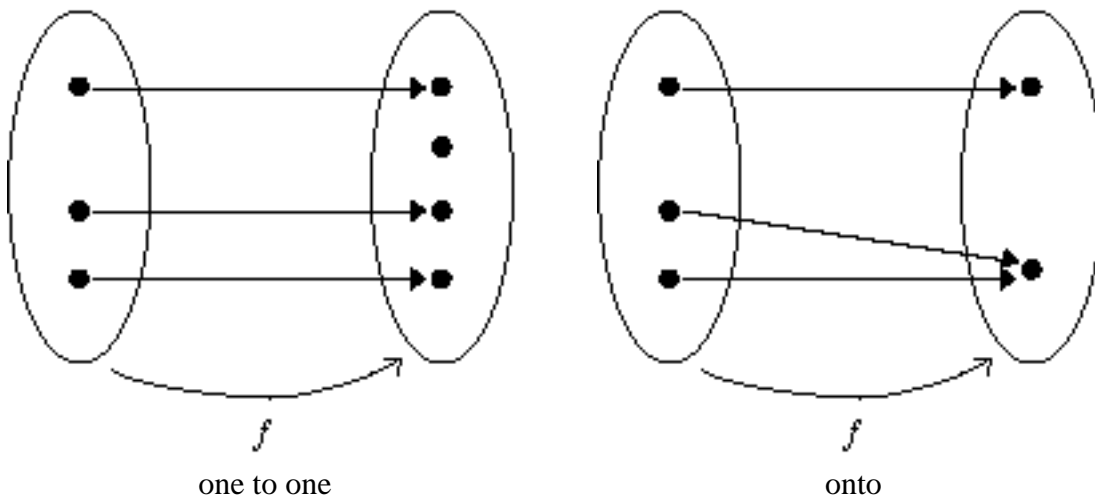
4.1.1 Definition. Let $G_1 = (P_1, L_1)$ and $G_2 = (P_2, L_2)$ be two abstract geometries, and let $f:$ $P_1 \to P_2$ a function that is bijective. Then we say that f is a *geometric transformation* if f also maps L_1 onto L_2.

In other words, a 1-1 transformation $f: P_1 \to P_2$ is *geometric* if takes the set P_1 of all points in G_1 onto the set P_2 of all points in G_2, and takes the set L_1 of all lines in G_1 onto the set L_2 of all lines in G_2. It is this last property that distinguishes geometric transformations from more general transformations. A more sophisticated way of formulating definition 4.1.1 is simply to say that $f: G_1 \to G_2$ is bijective. Notice that the definition makes good sense for models of both Euclidean and hyperbolic geometries. For instance, we shall see later that there is geometric bijection from the model $\mathbf{H}^2$ of hyperbolic geometry in terms of lines and planes in three space and the Poincaré disk model $\mathbf{D}$ in terms of points and arcs of circles.

Some simple examples from Euclidean plane geometry make the formalism much clearer. Let G_1 and G_2 both be models of Euclidean plane geometry so that P_1 and P_2 can be identified with all the points in the plane. For $f: P_1 \to P_2$ to be geometric it must map the plane onto itself, and do so in a 1-1 way, as well as map any straight line in the plane to a straight line. It will be important to see how such transformations can be described both algebraically and geometrically. It is easy to come up with functions mapping the plane onto itself, but it is much more restrictive for the function to map a straight line to a straight line. For example, $(x, y) \to (x, y^3)$ maps the plane onto itself, but it maps the straight line $y = x$ to the cubic $y = x^3$.

4.1.2 Examples. (a) Let

$$f : (x, y) \to (y, x)$$

be the function mapping any point $P = (x, y)$ in the plane to its reflection $P = (y, x)$ in the line $y = x$. Since successive reflections $P \to P \to P$ maps P back to itself, this mapping is 1-1 and maps the plane onto itself. But does it map a straight line to a straight line? Well the equation of a non-vertical straight line is $y = mx + b$. The mapping f interchanges x and y, so f maps the straight line $y = mx + b$ to the straight line $y = (x - b)/m$. Algebraically, f maps a non-vertical straight line to its *inverse*. Geometrically, f maps the graph of the straight line $y = mx + b$ to the graph of its straight line inverse $y = (x - b)/m$ as the figure below shows

y=mx+b

y=(x-b)/m

P

y=x

P'
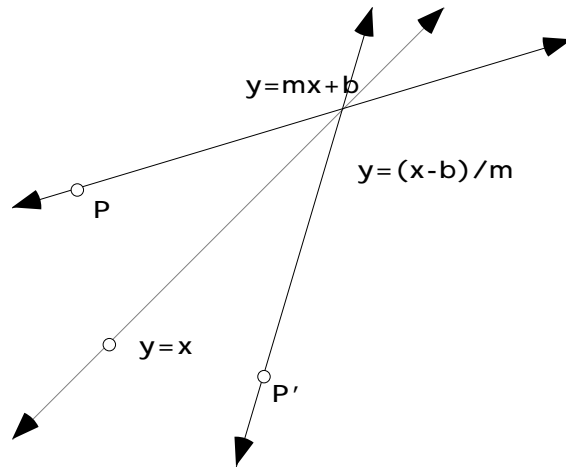
One can show also that *f* maps any vertical straight line to a horizontal straight line, and conversely. Hence *f* maps the family of all lines in Euclidean plane geometry onto itself - hence *f* is a geometric transformation of Euclidean plane geometry.

(b) More generally than in (a), given any fixed line *m*, let *f* be the mapping defined by *reflection* in the line *m*. In other words, *f* maps any point in the plane to its 'mirror image' with respect to the *mirror line m*. For instance, when *m* is the *x*-axis, then *f* takes the point $P = (x, y)$ in the plane to its mirror image $P = (x, -y)$ with respect to the *x*-axis. In general it is not so easy to express an arbitrary reflection in algebraic terms (see **Exercise Set 4.3**), but it is easy to do so in geometric terms. Given a point *P*, let *m* be the straight line through *P* that is perpendicular to *m*. Then *P* is the point on *m* on the opposite side of *m* to *P* that is equidistant from *m*. Again a figure makes this much clearer

m'

m

P

P'
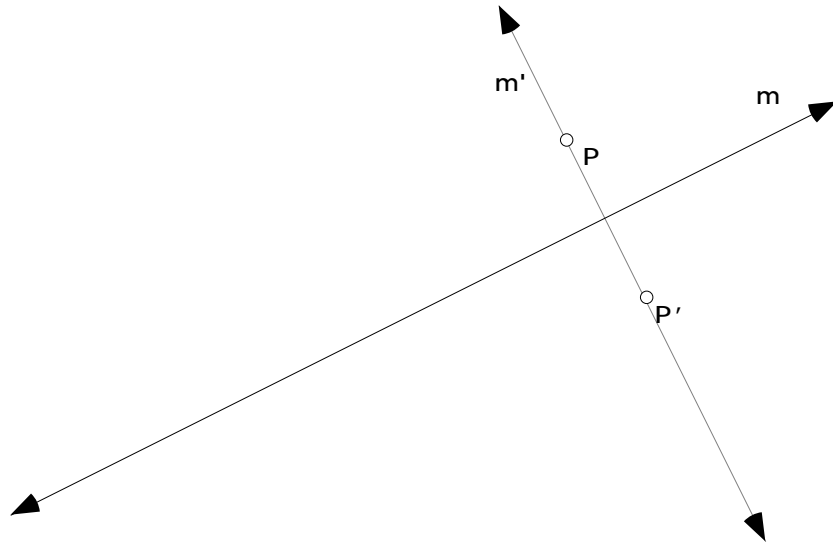
What is important to note here is that all these geometric notions make sense in hyperbolic geometry, so it makes good sense to define reflections in a hyperbolic line. This will be

done in Chapter 5 where we will see that this hyperbolic reflection can be interpreted in terms of the idea of inversion as hinted at in the last section of Chapter 3.

(c) Let f be a *rotation* through 90° counter-clockwise about the some fixed point in the plane. In algebraic terms, when the fixed point is the origin, f is given algebraically by $f : (x, y) \quad (-y, x)$. So f is 1-1 and maps the plane onto itself. What does f do to the straight line $y = mx + b$? (see **Exercise Set 4.3**)

(d) Let f be a *translation* of the plane in some direction. Then f is given algebraically by $f : (x, y) \quad (x + a, y + b)$ for some real numbers a and b. Again, it is clear that f is 1-1 and maps the plane onto itself.

Sketchpad is particularly useful for working with transformations because the basic transformations are all built into the program. We can use Sketchpad to look at the properties of reflections, rotations, and translations.

4.1.2a Demonstration.
- Open a new sketch on Sketchpad and draw a line. This will be the mirror line. Construct a polygon in the general shape of an "$\mathbb{F}$". Color its interior.
- To reflect the polygon across the mirror line, select the line and use the **Transform** menu to select "Mark Mirror". Under the **Edit** menu, select "Select All". Then under the **Transform** menu, select "Reflect".
- Try dragging some of the vertices of the polygon to investigate the properties of reflection in the mirror line. What happens when the mirror line is dragged?

Your figure should look like the following:
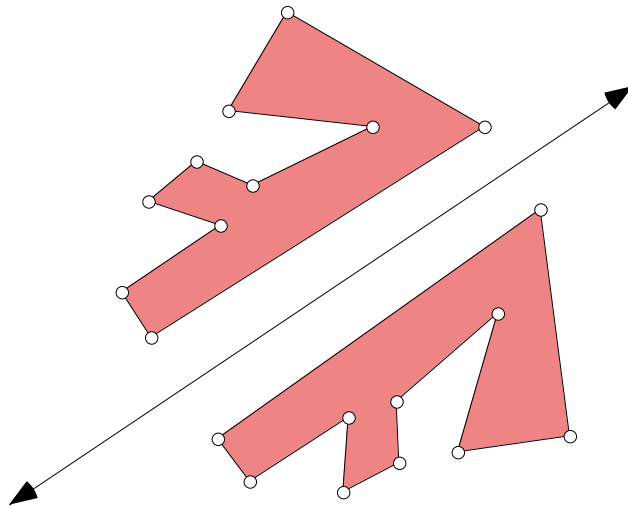
The *orientation* of the reflected "$\mathbb{F}$" is said to be *opposite* to that of the original "$\mathbb{F}$" because the clockwise order of the vertices of the image is the reverse of the clockwise order of the vertices of the pre-image. In other words, a reflection *reverses* orientation.

- Measure the area of each image polygon and its pre-image. Measure corresponding side lengths. Measure corresponding angles. Check what happens to your measurements as the vertices of the pre-image are dragged. What happens to the measurements when the mirror line is dragged? Now, complete **Conjecture 4.1.3.**

End of Demonstration 4.1.2a.

4.1.3 Conjecture. Reflections _____ distance, angle measure and area.

4.1.4 Definition. A geometric transformation *f* of the Euclidean plane is said to be an *isometry* when it preserves the distance between any pair of points in the plane. In other words, *f* is an isometry of the Euclidean plane, when the equality $d(f(a), f(b)) = d(a, b)$ holds for every pair of points a, b in the plane.

By using triangle congruences one can prove the following.

4.1.5 Lemma. Any isometry preserves angle measure.

The earlier Sketchpad activity supports the conjecture that every reflection of the Euclidean plane is an isometry. A proof of this can be given using congruence properties.

4.1.6 Theorem. Every reflection of the Euclidean plane is an isometry.

Proof. In the figure below P and Q are arbitrary points, while P' and Q' are their respective images with respect to reflection in the mirror line *m*. D and E are the intersection points between the mirror line and the segments $\overline{PP'}$ and $\overline{QQ'}$. For convenience we have assumed that P, Q lie on the same side of the mirror line. Use the definition of a reflection to show first that $\angle EDQ$ is congruent to $\angle EDQ'$, and hence that $\overline{DQ}$ is congruent to $\overline{DQ'}$. Now use this to show that $\angle PDQ$ is congruent to $\angle P'DQ'$. Hence $\overline{PQ}$ is congruent to $\overline{P'Q'}$.

<div align="right">**QED**</div>
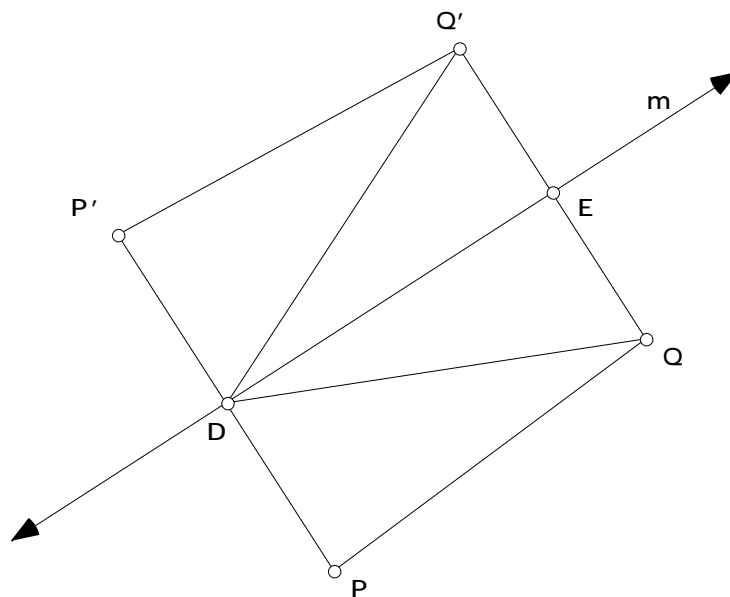
How would this proof have to be modified if P, Q lie on opposite sides of the mirror line? Notice by combining Lemma 4.1.5 with Theorem 4.1.6 we now have a proof of Conjecture 4.1.3.

Two other very familiar transformations of the Euclidean plane are rotations through a given angle about a given fixed point, and translation in a given direction by a fixed amount.

The most precise definition of these are terms of compositions of reflections (as we'll see in the next section), but direct geometric definitions can be given.
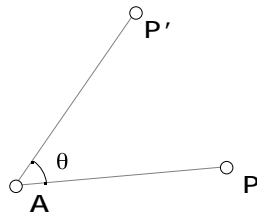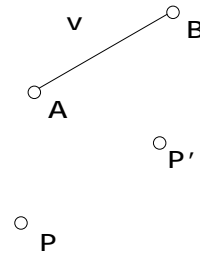
Rotation   Translation

Formally, a rotation $\rho_{A,\theta}$ about the point *A* through a directed angle θ is the transformation that fixes *A* and otherwise sends a point *P* to the point P' such that $\overline{AP'}$ is congruent to $\overline{AP}$ and θ is the directed angle measure of $\angle PAP'$. A translation T_v is the transformation that sends every point *P* the same distance direction, as determined by a given vector *v*. Again, Sketchpad makes the idea clear.

4.1.6a Demonstration.

- Open a new sketch and draw an "⊢".

- First we'll look at rotations. Construct a point and label it *A*. This will be the 'center' of the rotation, *i.e.*, the fixed point. Select the point *A* and then use the **Transform** menu to select "Mark Center A".

- Under the **Edit** menu, select "Select All". Then under the **Transform** menu select "Rotate". The rotate screen will pop up with the angle of rotation θ selected. You can change the degrees in a positive or negative direction.

- Investigate if rotation preserves distance, angle measure and area. Does rotation preserve or reverse orientation?

- Now for translations. Open a new sketch and draw an "⊢". Construct a line segment in a corner of your sketch and label the endpoints *A* and *B*. First select the endpoints in that order and the use the **Transform** menu to "Mark Vector A->B".

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- Using the Marquee (Arrow Tool) select the "𝔽". Under the **Transform** menu select "Translate". The translate screen will pop up with "By Marked Vector" selected. Click on "OK".

- Investigate if translation preserves distance, angle measure and area. Does rotation preserve or reverse orientation?   Now, complete **Conjecture 4.1.7.**

End of Demonstration 4.1.6a.

4.1.7 Conjecture. The rotation $\rho_{A,\theta}$ is _____ and also

_____ orientation. The translation $T_{A,B}$ is _____ and also

_____ orientation.

4.2 COMPOSITIONS. The usual composition of functions plays a very important role in the theory of transformations. Recall the general idea of composition of functions. Given functions f: A B and g: B C, mapping a set A into a set B and B into a set C respectively, then the *composition*

$$(g \circ f)(a) = g(f(a)), \qquad (a \quad A)$$

maps A into C. Pictorially, composition can be represented by the figure below
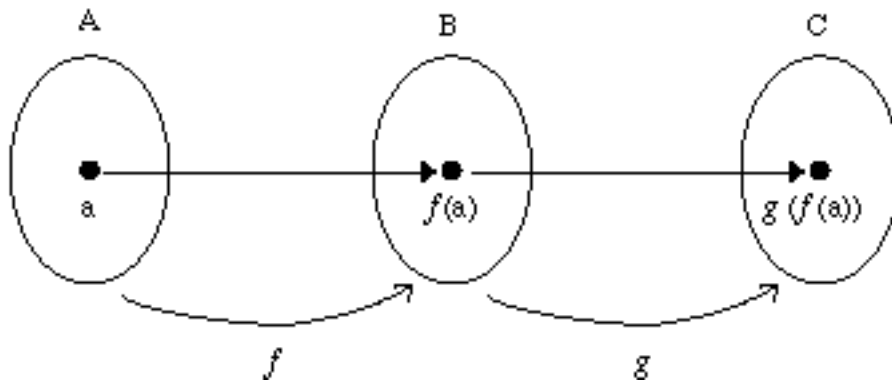
Notice that if f: A B and g: B C are bijective, then the composition will also be bijective.

4.2.1 Exercise. Show that if f: G_1 G_2 and g: G_2 G_3 are bijective, then the composition $g \circ f$ is bijective from G_1 onto G_3. In other words, the composition of geometric transformations is again geometric.

The concept of geometric transformation is very general. What we do is impose restrictions on a transformation $f: G_1 \quad G_2$ by imposing extra structure on G_1 and G_2 and then requiring that f *preserve* this extra structure. For instance, when a distance function is defined on G_1 and G_2 we can look only at geometric transformations $f: G_1 \quad G_2$ that preserve the distance between points - what we called isometries in the case of Euclidean geometry. If a notion of angle measure is defined on then we could look only at geometric transformations that preserve the angle between lines; such transformations are called *conformal* transformations. A complex-valued function $f:$ which is 1-1 and invertible on a set in the complex plane is conformal whenever f is analytic. This is one reason why analytic function theory is closely connected with geometry. (There are many interesting ideas for semester projects here if one knows something about complex numbers and analytic function theory.)

4.2.2 Theorem. Let f and g be isometric transformations of the Euclidean plane. Then the composition $g \circ f$ of f and g also is an isometric transformation of the Euclidean plane.

Proof. Let P and Q be arbitrary points in the plane. Since f is an isometry,
$$dist(P,Q) = dist(f(P), f(Q)).$$
But g also is an isometry, so
$$dist(f(P), f(Q)) = dist(g(f(P)), g(f(Q))).$$
Combining these two results we see that
$$dist(P,Q) = dist((g \circ f)(P), (g \circ f)(Q)).$$
Hence the composition $g \circ f$ preserves lengths and so is an isometry. **QED**

This theorem shows why there are close connections between geometry and group theory. For if $f: G \quad G$ is a geometric transformation, then f will have an inverse $f^{-1}: G \quad G$, and f^{-1} will be a geometric transformation; in addition, if f is an isometry, then f^{-1} will be an isometry. Thus the set of all geometric transformations $f: G \quad G$ is a group under composition, while the set of all isometries is a *subgroup* of this group. Now let's look more closely at the set of all isometries of the Euclidean plane - in more elaborate language, we are going to study the *Isometry Group* of the Euclidean plane. In the previous section we saw that any reflection is an isometry. Theorem 4.2.2 ensures that the composition of two reflections will be an isometry, and hence the composition of three, four or more reflections will be isometries as well. But how can we describe the composition of

reflections in geometric terms? Let's first use Sketchpad to see what happens for the composition of two reflections.

4.2.2a Demonstration. The Composition of Two Reflections.

- Open a new sketch and draw two mirror lines *l* and *l'*. Draw an "$\mathbb{F}$" somewhere in the plane.

- Now reflect this "$\mathbb{F}$" first in the mirror line *l* and then in the mirror line *l'*, producing a new image of "$\mathbb{F}$".

- Describe carefully the position of the final image "$\mathbb{F}$" in relation to the first "$\mathbb{F}$".

- What happens if the lines *l* and *l'* are parallel. What if they are not parallel?

- You should now be able to complete **Conjecture 4.2.3.**

End of Demonstration 4.2.2a.

4.2.3 Conjecture. The composition of reflections in two mirror lines is a
_____ when the mirror lines are parallel. The composition of reflections in two mirror lines is a _____ when the mirror lines intersect.

To investigate this more carefully, let's go once more to Sketchpad.

4.2.3a Demonstration.

- Open a new sketch and draw intersecting lines by first choosing three points *A*, *B*, and *C* then drawing two line segments *AB* and *AC*. The reason for constructing the mirror lines in this way is that dragging on *B* or *C* changes the angle between the mirror lines by rotating one of them about the vertex *A*.

- Now draw an "$\mathbb{F}$" on one side of a mirror line and then reflect it successively in the two mirror lines, producing a new image "$\mathbb{F}$" which should appear to be a rotation of the first "$\mathbb{F}$". Measure first the angle between the mirror lines and then measure the angle by line segments joining the vertex A to corresponding points on the first "$\mathbb{F}$" and its image. Compare the two values. This suggests that **Theorem 4.2.4** is true.

End of Demonstration 4.2.3a.

4.2.4 Theorem. Successive reflection in two intersecting mirror lines produces a rotation about the point of intersection through twice the angle between the mirror lines.

Proof. Consider the following figure, where P is first reflected in the mirror line AB with image P'. Then P' is reflected in the mirror line AC with image P''. There are two pairs of congruent triangles. By construction $PD = DP''$, so $\angle PAD$ is congruent to $\angle P'AD$ by the SAS criterion. Thus $\angle PAD = \angle P'AD$. By a similar argument $\angle P'AE = \angle P''AE$. Combining these two equalities we see that $\angle PAP'' = 2\angle DAE$. **QED**
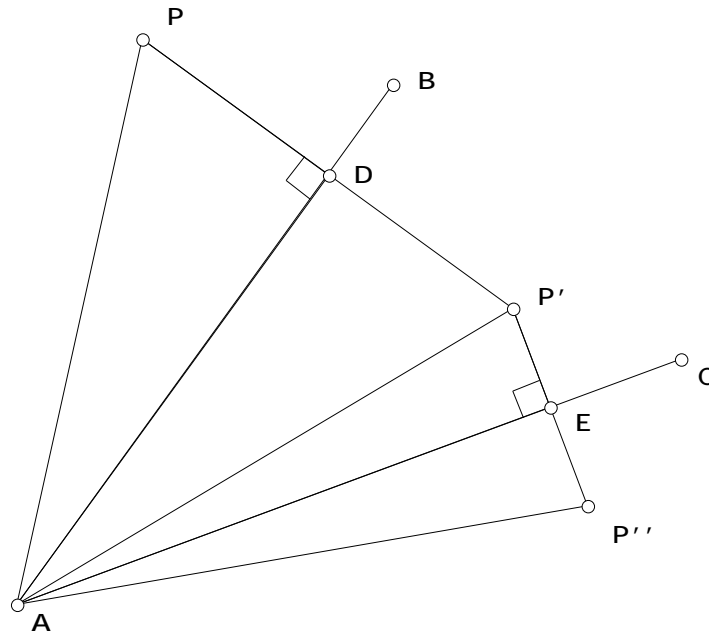
Now lets go back to Sketchpad and look at the case of parallel mirror lines.

- Open a new sketch and draw two parallel lines. On one side of these lines draw an "$\mathbb{F}$" and then reflect this successively in the two mirror lines. Drag one of the mirror lines so that it remains parallel to the other mirror line - you can do this by grabbing the line and then dragging. The image "$\mathbb{F}$"should then appear to be a translate of the first one.

- Measure the distance between the parallel mirror lines and then measure the distance between corresponding points on the first "$\mathbb{F}$" and the image "$\mathbb{F}$". Compare the two values.

4.2.5 Theorem. Successive reflection in parallel mirror lines produces a translation in a direction perpendicular to the mirrors through a distance equal to twice the distance between the mirrors.

Proof. See **Exercise Set 4.3.**

Next it makes sense to look at the composition of three reflections and see if we can describe the result in terms of rotations and translations as well. First we need to introduce one more Euclidean motion of the plane.

4.2.6 Definition. A glide reflection is the composition of a reflection with a translation parallel to the line of reflection.
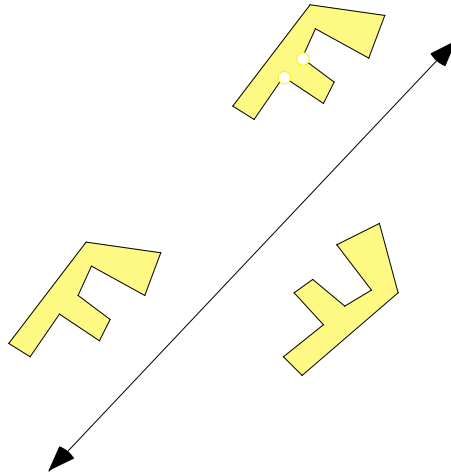
We should note that sketchpad does not have the glide reflection transformation built into the program. But we could easily build our own using scripts or custom transformations. We'll see how to use custom transformations in the next section.

A transformation in the plane has **direct orientation** if it preserves the orientation of any triangle. If the transformation does not preserve the orientation but reverses it then it has **opposite orientation**. Thus if a motion is the product of an even number of reflections then it will have direct orientation. If a motion is the product of an odd number of reflections then it will have opposite orientation. Rotations and translations are examples of _____ orientation while reflections and glide reflections show _____ orientation. This observation will help us when trying to describe the results of composing three reflections.

There are different cases that need to be considered when looking the possible outcomes of reflecting in three mirror lines.

4.2.6a Demonstration.

•   **Three Parallel Lines**: What do you get when you reflect something in three parallel lines? Draw three parallel lines and a simple polygonal figure. Reflect the figure successively about the 3 lines. (Hide intermediate figures to avoid confusion) What sort of transformation is this? What do the connected midpoints create? Draw at least 3 segments joining corresponding points on the pre-image to the final image. For each adjoining segment construct a midpoint and connect them together. Ignoring the three original lines what does this line suggest? How does your answer depend on the order of the lines? Investigate what happens when you change the order of reflection. (Drag the lines, say from #1 to #2)

•   **Two Parallel Lines and One Non Parallel**: What is this a composition of? Draw two parallel lines and one that crosses them both. Now draw a simple figure on the outside of the parallel line and below the transverse line. Reflect it about the parallel line, then again about the other parallel line. What kind of motion is this? Now reflect it in the transversal. What is this motion called and what is the result of the two combined? Does it make any difference where the figure ends if you reflect it in another sequence, say reflecting it in the transversal first? Does it matter if the transversal is perpendicular to the parallel lines?

•   **No Parallel Lines**: What sort of transformation does this case result in? Draw three lines that only intersect each other in one place. They should look like a triangle with its sides extended. Pick a place and draw yourself a small figure. Begin reflecting over the lines. What is the end result?

•   **Three Concurrent Lines**: What is the line of reflection for this case? To construct concurrent lines make sure the lines intersect at one point. Draw such lines. Draw a small figure between two of the lines. (It will be contained in a V shaped segment) Begin your reflections here. What sort of transformation is this? If you reflect a point all the way around the six lines what do you get? Start with a point where you had

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drawn your figure. Reflect it around each of the lines until you get back to the start. Is the last point is the same place as the first?

What happens if two of the mirror lines are identical? What happens if all three are identical?

You should be able to complete the following:

Product of Two Reflections	If the 2 lines of the reflection are parallel then the motion is a _____.
Product of Two Reflections	If the 2 lines of the reflection are not parallel then the motion is a _____.
Product of Three Reflections	If all 3 of the lines of the reflection are parallel then the motion is a _____.
Product of Three Reflections	If 2 of the lines of the reflection are parallel th the motion is a _____.
Product of Three Reflections	If the 3 lines of the reflection are concurrent then the motion is a _____.
Product of Three Reflections	If the 3 lines of reflection intersect each other only once then the motion is a _____.

End of Demonstration 4.2.6a.

With these notes in mind we can realize two of the most important theorems in the theory of isometric transformations of the Euclidean plane.

4.2.7 Theorem. Any isometry of the Euclidean plane can be written as a composition of no more than 3 reflections.

As a consequence of our exploration on composition of reflections we get the following as well.

4.2.8 Theorem. Any isometry of the Euclidean plane can be written as one of the following transformations: reflection, rotation, translation or glide reflection.

Crucial to the proof of Theorem 4.2.7 will be the following. To show if we are given an isometry and three points *A, B,* and *C* with image points *D, E,* and *F* we can take the composition of (at most) three reflections and also map *A, B,* and *C* to *D, E,* and *F* respectively. If the orientation of the points is preserved it will take two reflections, and otherwise it will take three reflections.

- Open the "composition3ref" sketch. Notice △*ABC* can be mapped to △*DEF* by a glide reflection. The midpoints of the segments joining pre-image points to their corresponding image points define the line of reflection. Then the final image is a translation of the reflected image. We will show we can also map △*ABC* to its image by the composition of three reflections.
- Draw a line segment between A and D and find the midpoint. Construct the line *l* perpendicular to the line segment and through the midpoint. Reflect △*ABC* in *l* and A will be mapped onto D. So there is one point in the correct position and one reflection.
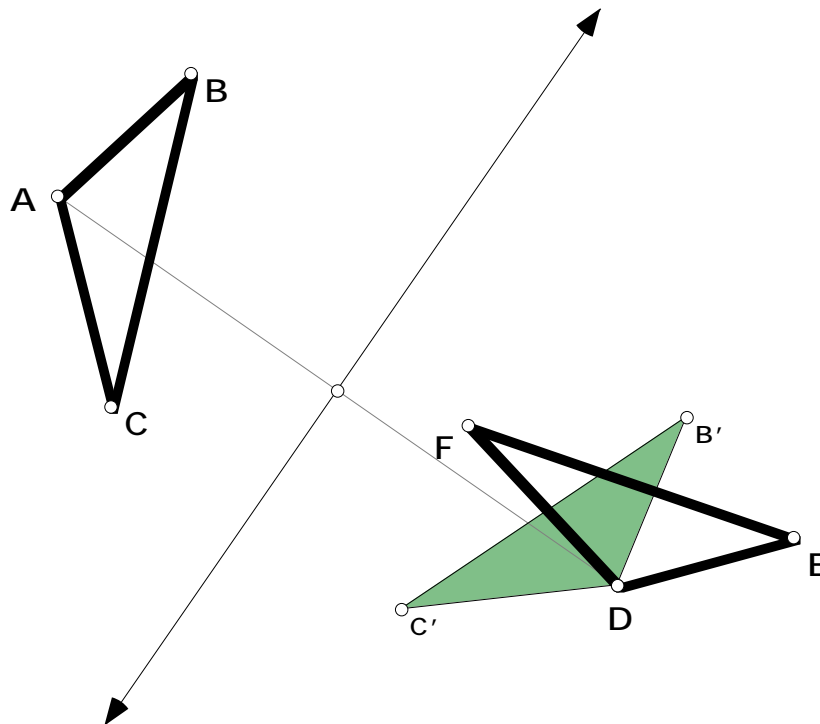
- If B and C also land on E and F then you would be done. If this is not this case, then we are to map B́ to E by reflecting through the perpendicular bisector of $\overline{B'E}$ where

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B' is the image of B under the first reflection. This maps B' to E and keeps D fixed. Why does D stay fixed?
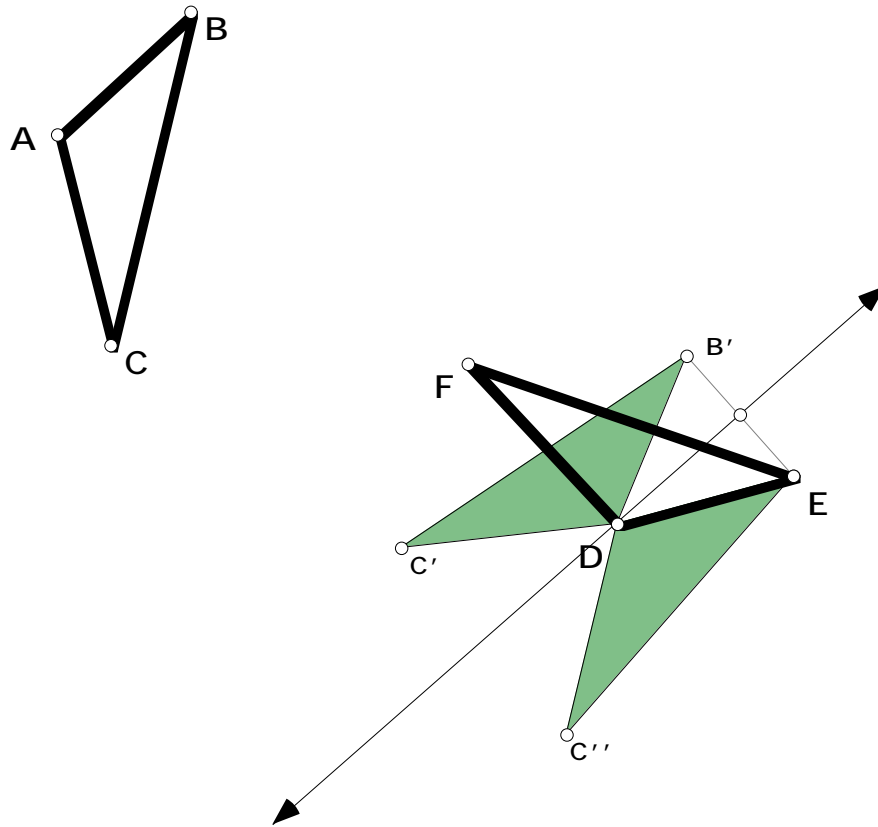
- This leaves you with only *C''* (from the original *C*) to be mapped. If it falls on *F* after the second reflection then you would be done, but if it does not, map *C''* to map to *F* by reflecting about the line $\overrightarrow{DE}$. Why is $\overrightarrow{DE}$ the perpendicular bisector of $\overline{FC}$? Now you are done and it has taken 3 reflections to get from the pre-image to the final image.

Before proving Theorem 4.2.7 we need to establish another property of isometries.

4.2.9 Lemma. An isometry maps any three non-collinear points into non-collinear points.

Proof. Let *A*, *B*, and *C* be non-collinear points. Then by the triangle inequality the non-collinearity means that

$$dist(A, B) + dist(B, C) > dist(A, C).$$

Now let A', B', and C' be the images of A, B, and C. Since the isometry preserves distances,

$$dist(A', B') + dist(B', C') > dist(A', C').$$

But this ensures that A', B', and C' cannot be collinear, proving the lemma.   **QED**

Proof of Theorem 4.2.7. Given an isometry F, choose a set of non-collinear points A, B, and C. Let $A' = F(A)$, $B' = F(B)$, and $C' = F(C)$ be their images. Suppose that F has preserved orientation of ABC. Then the Sketchpad activity on 'Composition of reflections' shows that there exist reflections S_1 and S_2 so that their composition $S_1 \circ S_2$ has the properties

$$(S_1 \circ S_2)(A) = A' \ , \ (S_1 \circ S_2)(B) = B' \ , \ (S_1 \circ S_2)(C) = C' \ .$$

We will prove that

$$(S_1 \circ S_2)(P) = F(P).$$

holds for every point P. So set

$$(S_1 \circ S_2)(P) = P' \ , \ F(P) = P'' \ .$$

We have to show that $P' = P''$. Because $S_1 \circ S_2$ and F are isometries,

$$dist(A', P') = dist(A', P''), dist(B', P') = dist(B', P''), dist(C', P') = dist(C', P'').$$

Thus A', B', and C' will all lie on the perpendicular bisector of the segment $\overline{P' P''}$ if $P' \neq P''$. But this can happen only if A', B', and C' are collinear. But A, B, and C are not collinear, so A', B', and C' are not collinear. Hence $P' = P''$ showing that $S_1 \circ S_2 = F$. If F does not preserve the orientation of ABC then the same proof will show that F can be

written as the composition of either one reflection or three reflections. This completes the proof. **QED**

4.3 Exercises. The problems in this assignment are a combination of algebraic and geometric ones.

Exercise 4.3.1. Show that the function $f : (x, y) \quad (-y, x)$ maps the straight line $y = mx + b$ to the straight line $y = -(x + b)/m$. Explain the relationship between the slopes of these two lines in terms of the transformation in 4.1.2 (c).

Exercise 4.3.2. Show that reflection in the line $y = mx$ is given by

$$f : (x, y) \qquad \frac{2m}{m^2 + 1} \; y - \frac{m^2 - 1}{m^2 + 1} \; x, \; \frac{2m}{m^2 + 1} \; x + \frac{m^2 - 1}{m^2 + 1} \; y \; .$$

Hint: Let the reflection of the point $P = (x, y)$ be $P' = (x', y')$. You need to find two equations and then solve for x', y'. Let Q be the midpoint of PP'; so what are its coordinates? The point Q also lies on the mirror line $y = mx$; so what does this say about the coordinates of Q'? Use this to get the first equation for x', y'. The line PP' is perpendicular to the mirror line $= mx$. How can we use this to get a second equation for x', y'? Now solve the two equations you have obtained.

Exercise 4.3.3. Prove synthetically that every rotation $\rho_{A,\theta}$ is an isometry.

Exercise 4.3.4. Prove that successive reflections in parallel mirror lines produce a translation in a direction perpendicular to the mirrors through a distance equal to twice the distance between the mirrors.

Exercise 4.3.5. Suppose you wish to join the two towns $A(1,5)$ and $B(8,2)$ via a pipeline. A pumping station is to be placed along a straight river bank (the x-axis). Determine the location of a pumping station, $P(x,0)$, that minimizes the amount of pipe used? Solve this
- by transformations.
- by calculus.

A(1,5)

B(8,2)

P(x,O)
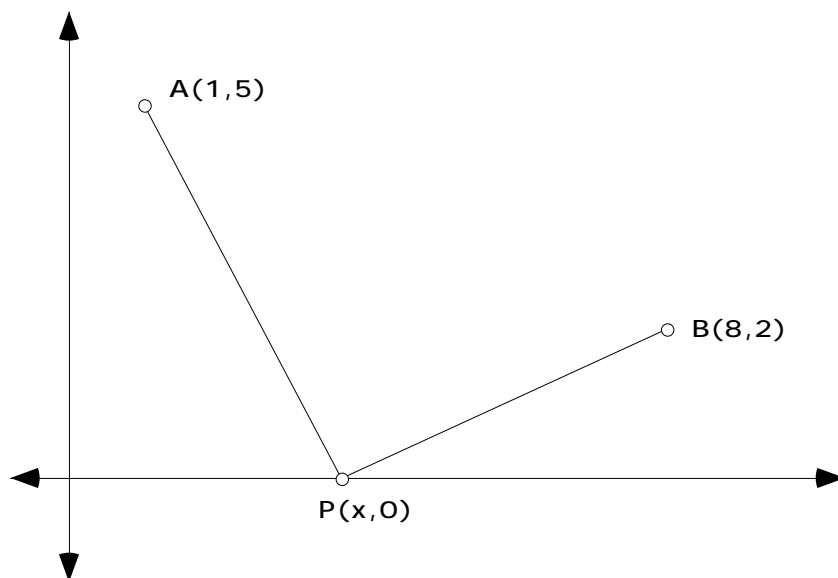
Exercise 4.3.6 Buried Treasure. Among his great-grandfather's papers, José found a parchment describing the location of a hidden treasure. The treasure was buried by a band of pirates on a deserted island which contained an oak tree, a pine tree, and a gallows where the pirates hanged traitors. The map looked like the accompanying figure and gave the following directions.

"Count the steps from the gallows to the oak tree. At the oak, turn 90° to the right. Take the same number of steps and then put a spike in the ground. Next, return to the gallows and walk to the pine tree, counting the number of steps. At the pine tree, turn 90° to the left, take the same number of steps, and then put another spike in the ground. The treasure is buried halfway between the spikes."

José found the island and the trees but could not find the gallows or the spikes, which had long since rotted. José dug all over the island, but because the island was large, he gave up. Devise a plan to help José find the treasure.
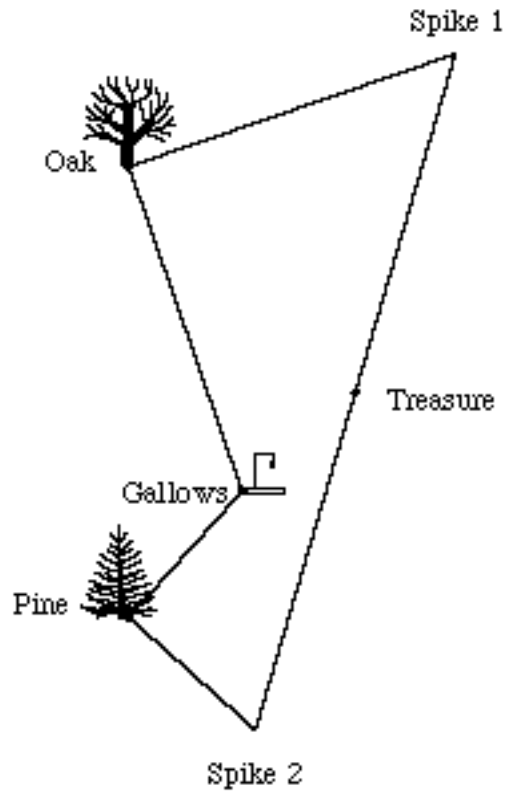
4.4 TILINGS REVISITED. To illustrate further the idea of reflections, rotations, translations, and glide reflections we want to begin the geometric analysis of 'wallpaper' designs. A wallpaper design is a tiling of the plane that admits translational symmetry in two directions. That is the design can be "moved" in two different directions and coincide with itself. The checkerboard below would produce a wallpaper design if continued indefinitely.
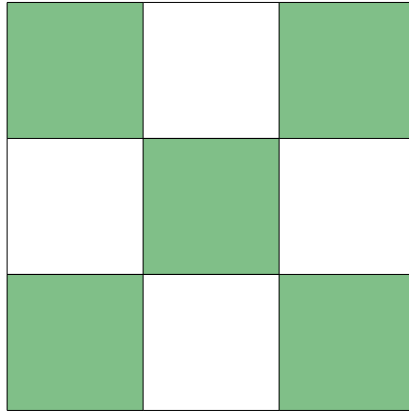
First we notice that certain rotations are admissible. For the checkerboard we can rotate by $90°$ (quarter-turn) about the center of any green or white square and repeat the same figure. Also we can rotate by $180°$ (half-turn) about the vertex of any square and repeat the same figure. There are wallpaper designs that admit $60°$ (sixth-turn) rotations and $120°$ (third-turn) rotations. What is more remarkable is that these are the only rotations allowed in any wallpaper design! A simple argument shows why. (See Crowe) To get you started on the fifth-turn case, try the following. Choose one center of rotation P and then choose another center of rotation that is closest to Q. Next argue why this cannot happen. The n-th turn case is even easier.

This restriction on rotations provides a convenient way to analyze wallpaper patterns. In fact, it can be shown that there are only 17 different types of wallpaper designs!

- Four which have no rotations at all;
- Five whose smallest rotation is 180°;
- Three whose smallest rotation is 120°;
- Three whose smallest rotation is 90°;
- Two whose smallest rotation is 60°;

There is a simple flowchart one can use to classify any wallpaper design. The symbols for the patterns have special meaning: *m* means mirror, *g* means glide, and a number like 2 or 4 means half-turn or quarter-turn.