Lie Groups Solutions, Problem Set # 7

Section 4.2:

6: (a) A linear transformation $M \in GL(E)$ sends subspaces of E to subspaces of E. It is clear that this action is transitive on $Gr_m(E)$, making $Gr_m(E)$ a homogeneous space for $GL(E)$. The only question is what the stabilizer of a a point in $Gr_m(E)$ is. Let $P_0 \in Gr_m(E)$ be the subspace spanned by the first m (standard) basis vectors. A matrix that sends P_0 to itself must send each of these basis vectors to a vector in P_0. It can send the remaining $n-m$ standard basis vectors to anything (as long as the matrix is invertible). That is, the matrix must take the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

Now, if $P \in Gr_m(E)$ is a different subspace, then the stabilizer of P is conjugate to the stabilizer of P_0 by a matrix that sends P_0 to P. But that's just a change-of-basis matrix to a new basis whose first m vectors are a basis for P. (BTW, this SHOWS that that action of $GL(E)$ is transitive on $Gr_m(E)$.)

(b) Let P_0 be as before, the subspace whose basis is the first m standard basis vectors, and let P be another subspace of E of the same dimension. Pick an orthonormal basis for P, and extend this to an orthonormal basis for E. The matrix whose columns are these basis vectors will lie in $K(E)$, and will send P_0 to P. This shows that $K(E)$ acts transitively on $Gr_m(E)$, hence that $Gr_m(E)$ is a homogeneous space for $K(E)$. As before, a matrix $K(E)$ that sends P_0 to itself must take the form $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$. Since $M^*M = 1$, we must have $A^*A = 1$, $B^*B + C^*C = 1$. In particular, $A^* = A^{-1}$. Since $MM^* = 1$, $AA^* + BB^* = 1$ and $CC^* = 1$. But $AA^* = 1$, so $BB^* = 0$, so $B = 0$. So we are left with matrices of the form $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$ with A and C unitary (or, in the real case, orthogonal).

That is $Gr_m(E) = U(n)/(U(m) \times U(n-m))$ for complex E, and $Gr_m(E) = O(n)/(O(m) \times O(n-m))$ for real E.

9: Since H is a connected closed subgroup of G, $H \subset G_0$, so we can take the quotient G_0/H. This set is open (since G_0 is open) and closed (since G_0 and H are closed). Since G/H is connected, this implies that $G_0/H = G/H$. I claim this implies that $G_0 = G$. For suppose that $x \in G$. Then $xH = yH$ for some $y \in G_0$, so $x = yh_1$ for some $h_1 \in H \subset G_0$. But $yh_1 \in G_0G_0 = G_0$.

10: a) Since $S^{n-1} = SO(n)/SO(n-1)$ is connected, and since $SO(2) = S^1$ is connected, it follows by induction on n that $SO(n)$ is connected (using the result of problem 9). Note that this only works for $n \geq 2$. The case of $SO(1) = 1$ is separate.

b) $SL(n, \mathbf{R})$ acts transitively on $\mathbf{R}^n - \{0\}$, and the stabilizer of the point $(1, 0, 0, \ldots, 0)^T$

is all matrices of the form $\begin{pmatrix} 1 & A \\ 0 & B \end{pmatrix}$, where $A \in \mathbf{R}^{n-1}$ is a row vector and $B \in SL(n-1, \mathbf{R})$. Since $\mathbf{R}^{n-1}$ is connected, we can apply our induction argument. $SL(1, \mathbf{R}) = \{1\}$ is connected. If $SL(n-1, \mathbf{R})$ is connected, then $H = SL(n-1, \mathbf{R}) \times \mathbf{R}^{n-1}$ is connected, and $R^n - \{0\} = SL(n, \mathbf{R})/H$ is connected, so $SL(n, \mathbf{R})$ is connected.

Section 4.3:

5: To get the action of $\exp(t \text{ Drive})$, we first note that θ is constant and $\phi(t) = \phi(0) + t\sin(\theta)$. We then integrate the changes in x and y to get that $\exp(t \text{ Drive})(x_0, y_0, \phi_0, \theta_0) = (x_0 + [\sin(\phi_0 + \theta_0 + t\sin(\theta_0)) - \sin(\phi_0 + \theta_0)]/\sin(\theta_0), y_0 - [\cos(\phi_0 + \theta_0 + t\sin(\theta_0)) - \cos(\phi_0 + \theta_0)]/\sin(\theta_0), \phi_0 + t\sin(\theta_0), \theta_0)$.

Integrating Steer is trivial: $\theta \to \theta + t$, with all other coordinates constant.

Integrating Wriggle is similar to integrating Drive, since the formulas are IDENTICAL up to the substitution $\theta \to \theta - \pi/2$. The result is that $(x_0, y_0, \phi_0, \theta_0)$ goes to $(x_0 + [\cos(\phi_0 + \theta_0 + t\cos(\theta_0)) - \cos(\phi_0 + \theta_0)]/\cos(\theta_0), y_0 + [\sin(\phi_0 + \theta_0 + t\cos(\theta_0)) - \sin(\phi_0 + \theta_0)]/\cos(\theta_0), \phi_0 + t\cos(\theta_0), \theta_0)$.

Integrating Slide sends $(x_0, y_0, \phi_0, \theta_0)$ to $(x_0 - t\sin(\phi_0), y_0 + t\cos(\phi_0), \phi_0, \theta_0)$.

6: Take Steer $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, Drive $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,
Wriggle $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ and finally Slide $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}$. You can check that all of the commutation relations work out correctly.