PRELIMINARY EXAMINATION IN ALGEBRA
January 2000 4 Hours (1:00 - 5:00)

Directions: Do three out of the four questions below and please indicate here which questions you chose:

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1. Let $f(x) = x^3 - 7$ in each part below.
   a) Let K be the splitting field of $f(x)$ over $\mathbb{Q}$, the field of rational numbers. Describe the Galois group $\mathrm{Gal}(K/\mathbb{Q})$, and describe the intermediate fields between K and $\mathbb{Q}$. Which intermediate fields are not Galois over $\mathbb{Q}$?
   b) Let L be the splitting field of $f(x)$ over $\mathbb{R}$, the field of real numbers. Describe the Galois group $\mathrm{Gal}(L/\mathbb{R})$.
   c) Let M be the splitting field of $f(x)$ over $\mathbb{F}_{13}$, the finite field with thirteen elements. Describe the Galois group $\mathrm{Gal}(M/\mathbb{F}_{13})$.

2. What is the order of $GL_3(\mathbb{Z}/4\mathbb{Z})$, the group of 3×3-invertible matrices over $\mathbb{Z}/4\mathbb{Z}$?

3. Let G be a non-abelian group of order p^3, p prime. Prove that the center $Z(G)$ of G is of order p and that $Z(G) = [G, G]$ the commutator subroup of G, that is the group generated by $xyx^{-1}y^{-1}, x, y \in G$.

4. Let R be a PID and F its field of fractions. Suppose S is a ring with $R \subset S \subset F$.
   a) Show that all elements $\alpha \in S$ can be written as a/b, where $a, b \in R$ and $1/b \in S$.
   b) Show that S is a PID.
   c) Show that if S is finitely generated as an R-module then $S = R$.