Preliminary Examination in Algebra
August 21, 2001, RLM 9.166, 1:00-5:00 p.m.

Do three of the following four problems.

1. Suppose that k is a field of characteristic $p > 0$.

   (i) If a is an element of k, show that the polynomial $f(x) = x^p - a$ either splits in $k[x]$ or is irreducible in $k[x]$.

   (ii) Let $\sigma : k \to k$ be the Frobenius endomorphism defined by $\sigma(\alpha) = \alpha^p$. Assume that σ is an automorphism, that is, assume that σ is surjective. Prove that every irreducible polynomial in $k[x]$ is separable.

2. Let G be a finite group.

   (i) Suppose that G has order n and p is the smallest prime divisor of n. Show that any subgroup of index p is normal.

   (ii) Suppose that G has order 255. Prove that G has a normal subgroup of order 17, and prove that G has a cyclic normal subgroup of order 85. From these facts deduce that any group of order 255 is cyclic.

3. Let R be a principal ideal domain, M a finitely generated free R-module and $S \subset M$ a submodule. Show that the following are equivalent. (State clearly any theorems you use about modules over a PID.)

   (a) The submodule S is *complemented*, that is, there exists a submodule $T \subset M$ with
   $$M \cong S \oplus T.$$

   (b) The quotient module M/S is a free R-module.

   (c) If $x \in S$ and $x = ay$ for some $y \in M$, $a \in R$ ($a \neq 0$), then $y \in S$.

4. Let $\mathbb{Q}$ denote the field of rational numbers.

   (i) Find a Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}/3\mathbb{Z}$ (the cyclic group of order three).

   (ii) Find a Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.