Algebra Preliminary Examination

January 2001

Work three of the four problems. Indicate clearly which three problems you are working.

1. Let $|S|$ denote the cardinality of a finite set S. Let G be a finite group, and let C be a subgroup of G.

   (a) Suppose K is a normal subgroup of G. Suppose p is a prime number which DOES divide $|C|$ but does NOT divide $|C \cap K|$. Show p DOES divide $|G/K|$.

   (b) Let G' be the commutator subgroup of G. Suppose p is a prime number which divides $|C|$ but does not divide $|C \cap G'|$. Show G has a subgroup of index p.

   Now let K be a normal subgroup of G, and let p be a prime number which divides $|K|$. Let $Sp(G)$ (respectively, $Sp(K)$) denote the set of Sylow p-subgroups of G (respectively, K).

   (c) Show $Sp(K) = \{P \cap K \mid P \in Sp(G)\}$.

   (d) Show $|\,Sp(K)|$ divides $|\,Sp(G)|$.

2. Let $\mathbb{Z}[x]$ denote the ring of polynomials in one variable with integer coefficients.

   (a) Describe all maximal ideals in $\mathbb{Z}[x]$ by giving generators for the maximal ideals, and explain why your answer is correct.

   If R is a commutative ring with identity element 1, recall that an ideal P of R with $P \neq R$ is called a PRIME IDEAL if for every $a, b \in R$,

$$ab \in P \Longrightarrow a \in P \quad \text{or} \quad b \in P\ .$$

   (b) Show that the ideal $(3, x^2 + x + 1)$ in $\mathbb{Z}[x]$ is not a prime ideal by exhibiting $f(x), g(x) \in \mathbb{Z}[x]$ with $f(x)g(x) \in (3, x^2 + x + 1)$ but with $f(x) \notin (3, x^2 + x + 1)$ and $g(x) \notin (3, x^2 + x + 1)$.

   (c) Is every nonzero prime ideal in $\mathbb{Z}[x]$ a maximal ideal? If yes, prove it. If no, give an example (with proof) of a nonzero prime ideal which is not maximal.

3. Let $f(x)$ be a separable irreducible monic polynomial over a field F, and let K be the splitting field of $f(x)$ over F.

Suppose deg $f(x) = p$, where p is a prime number.

Let G be the Galois group of K over F.

Let P be a Sylow p-subgroup of G.

Let N be the normalizer of P in G.

SUPPOSE $P = N$. (The goal of this problem is to show $P = G$.)

(a) Show that $|G| = ps$ with s an integer not divisible by p.

(b) With s as in part (a), show there are exactly $s(p-1)$ elements of order p in G. (Hint: How many conjugates does P have in G?)

Now let α and β be roots of $f(x)$ in K. Let H_α be the Galois group of K over $F[\alpha]$, and let H_β be the Galois group of K over $F[\beta]$.

(c) Show G is the disjoint union of H_α and $\{\sigma \in G \mid \text{ order of } \sigma \text{ is } p\}$. (Hint: Compute $|H_\alpha|$ and $|G| - |H_\alpha|$.)

(d) Show $H_\beta = H_\alpha$ and $F[\beta] = F[\alpha]$.

(e) Show $K = F[\alpha]$, and conclude $P = G$.

4. (a) Let $\mathbb{C}^{n\times n}$ denote the set of $n \times n$ matrices with entries in $\mathbb{C}$, and let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$. Prove there exists a nonsingular $T \in \mathbb{C}^{n\times n}$ such that TAT^{-1} and TBT^{-1} are diagonal (i.e., A and B can be diagonalized simultaneously) if and only if $AB = BA$.

(b) Let k be a field, and let $A \in k^{n\times n}$ be nonsingular. Prove that there exists a polynomial $q(x) \in k[x]$ such that $A^{-1} = q(A)$.