Algebra Prelim , August 2002

Do three out of the four questions below.

1. Let p and q be distinct primes and G a finite group with p^3q elements.

a) Prove that G has either a normal p-Sylow subgroup or a normal q-Sylow subgroup unless $p = 2, q = 3$.

b) Prove that the symmetric group S_4 does not have a normal 2-Sylow subgroup and does not have a normal 3-Sylow subgroup.

2. Prove or give a counterexample to each of the statements below, where thoroughout the problem it is assumed that R and S are integral domains and $f : R \to S$ is a surjective homomorphism.

a) If R is a PID then S is a PID.

b) If R is a UFD then S is a UFD.

c) If R is a PID and f is not an isomorphism then S is a field.

d) If R is a UFD and f is not an isomorphism then S is a field.

3. One defines a division ring to be a (not necessarily commutative) ring in which every non-zero element has a multiplicative inverse. Let D be a finite division ring. Let $Z = \{x \in D \mid xy = yx, \forall y \in D\}$ be the center of D.

a) Prove that Z is a finite field. If $Z = \mathbf{F}_q$, show that D has q^n elements for some n.

b) Show that, given $x \in D$, the set $Z(x) = \{y \in D \mid xy = yx\}$ is a division ring containing Z and conclude that $Z(x)$ has q^{n_x} elements for some n_x.

c) Show that $D^* = D \setminus \{0\}$ is a group under multiplication. Show that the centralizer of $x \in D^*$ is $Z(x) \setminus \{0\}$. Apply the class equation to D^* to conclude that $q^n - 1 = q - 1 + \sum \frac{q^n-1}{q^{n_i}-1}$ for some integers n_i.

4. Let F be a field and K/F a finite extension of degree $[K : F] = n$. Let $f(x) \in F[x]$

be an irreducible polynomial of degree m. Show that if n and m are relatively prime then $f(x)$ remains irreducible in $K[x]$.