Algebra Prelim August 2003

Do any three problems. If you write something about more than three problems, mark which three you want graded. You have four hours.

#1. Let G be the group of 2 by 2 matrices of determinant one over the field, F_3, of three elements. Group theorists usually write $G = SL_2(3)$. The group operation is matrix multiplication, but when we add elements we are doing it in the full 2 by 2 matrix ring $M_2(F_3)$.

a) If $g \in G$, show that g satisfies the polynomial equation $x^2 - ax + 1 = 0$ for some $a \in F_3$. Factor, or prove irreducible, each of these three polynomials in $F_3[x]$.

b) If g satisfies $x^2 + 1 = 0$, show that g has order 4. If g satisfies $x^2 + x + 1 = 0$, show that g has order 1 or 3. If g satisfies $x^2 - x + 1 = 0$, show that g has order 2 or 6. (Hint: canonical forms) Show that there is a unique element of order 2 in G.

c) Let $V = F_3 \oplus F_3$ and view $M_3(F_3)$ as $\mathrm{End}_{F_3}(V)$. Show that G has order 24. If $g \in G$ has order 3, show that the set of $v \in V$ such that $g(v) = v$ is a one dimensional subspace of V.

d) Let $v \in V$ be nonzero and $H \subset G$ the subgroup of $g \in G$ with $g(v) = v$. Show that H has order 6 and has a normal subgroup of order 3. Use this and c) to show G has 4 subgroups of order 3.

e) Let $\iota \in G$ be the element of order 2, and set $\bar{G} = G/ < \iota >$. Show that $\bar{G}$ also has 4 subgroups of order 3, and then show $\bar{G}$ has a normal subgroup of order 4. Show G has a normal subgroup of order 8.

#2. Let K be a subfield of the complex field $\mathbb{C}$. Let $f(x) \in K[x]$ be a polynomial of degree n. Let G be the Galois group of of the splitting field, $L \subset \mathbb{C}$, of $f(x)$ over K.

a) Suppose $\alpha \in \mathbb{C}$ is a root of $f(x)$ with the following property. If $\beta \in \mathbb{C}$ is another root, there is a polynomial $g_\beta(x) \in K[x]$ with $g_\beta(\alpha) = \beta$. Assuming $f(x)$ irreducible, what can you say about the order of the group G? For general $f(x)$, what can you say?

b) In 1829 Abel considered polynomials with the additional property that for all roots β, γ, of $f(x)$, $g_\beta(g_\gamma(\alpha)) = g_\gamma(g_\beta(\alpha))$. In 1877 Kronecker published a paper calling these polynomials "Abelschen". What special property of G does this property imply?

c) Show, by example, that for $K = \mathbb{Q}$ there exists polynomials of arbitrarily high degree with the property in part b).

d) Assume the condition in a), and suppose G is a p group. Show that there is a root $\beta \neq \alpha$ such that $g_\beta(g_\gamma(\alpha)) = g_\gamma(g_\beta(\alpha))$ for each root γ of $f(x)$.

#3. Let R be a commutative ring.

a) Define what it means for R to be Noetherian.

b) Prove that if R is Noetherian, then the polynomial ring $R[x]$ is Noetherian.

#4. Let R be the ring of continuous real functions $\mathbb{R} \to \mathbb{R}$ such that $f(x + \pi) = f(x)$ for all $x \in \mathbb{R}$. Let M be the set of continuous functions $u : \mathbb{R} \to \mathbb{R}$ such that $u(x+\pi) = -u(x)$

for all $x \in \mathbb{R}$. Note that with the usual addition and multiplication of functions M is an R module (you need not prove this). Let $c, s \in M$ be the usual cosine and sine functions.

a) Show that $u \in M$ implies that $u(a) = 0$ for some $a \in \mathbb{R}$. (Yes, you can use calculus.)

b) Show that M is not generated by one element. (Hint: Use a).)

c) Show that the map $(f, g) \mapsto (fc + gs, -fs + gc)$ is an isomorphism $R \oplus R \cong M \oplus M$. Show that M is projective as an R module.

d) Show that $f \mapsto fs \otimes s + fc \otimes c$ is an isomorphism $R \cong M \otimes_R M$. (Hint: Find an inverse.)

#5. Let G be a finite group of order 90 with a non-normal subgroup, H, of order 5.

a) Show that G has a subgroup of order 15. Show this subgroup is abelian.

b) Let g be an element of order 3 in the subgroup of order 15 in a). Show that the centralizer of g has order 90 or 45. In the later case, show G has a normal subgroup of order 5, a contradiction.

c) Suppose the element g of part b) has centralizer the whole group G. Form the quotient $\bar{G} = G/ < g >$. Show the image of H is normal in $\bar{G}$.

d) Conclude that G has a normal subgroup of order 15. Show G has a normal subgroup of order 5, another contradiction.

Okay, so any group of order 90 has a normal subgroup of order 5.

e) Show that such a G has a normal abelian subgroup of order 45.

#6. Let F be a field of characteristic 0. Suppose V is a finite dimensional vector space over F, and $A \in \text{End}_F(V)$ is such that $A^2 = A$.

a) Show that V is the internal direct sum of the kernel and image of A.

b) Show that the following three properties are equivalent:

i) $A \neq 0$

ii) There is a nonzero $v \in V$ with $Av = v$.

iii) The trace of A is nonzero.

c) Suppose V is the internal direct sum of subspaces W_1 and W_2. Show that there is an A as above with kernel W_1 and image W_2.

d) Suppose instead that F has nonzero characteristic p. Which of the conditions in b) is no longer equivalent to the others? Give an example.