

**Preliminary Examination in Algebra**  
**January 7, 2003, RLM 9.166, 1:00-5:00 p.m.**

Do three of the following four problems.

1. Assume that  $G$  be a finite group,  $p$  the smallest prime dividing the order of  $G$ , and  $H$  a subgroup of index  $p$  in  $G$ .
  - (i) Prove that  $H$  is a normal subgroup of  $G$ .
  - (ii) Show that if  $p + 2$  is a prime and  $G$  is of order  $p(p + 2)$ , then  $G$  is cyclic.
  
2. Let  $R$  be an integral domain and let  $a$  be a non-zero non-invertible element of  $R$ .
  - (i) Prove that the ideal  $(a, X) \subset R[X]$  is not principal.
  - (ii) Use (i) to show that, if  $K$  is a field, then  $K[X, Y]$  is not a PID.
  
3. Let  $D$  be a unique factorization domain and let  $K$  be its field of fractions.
  - (i) Prove that  $D$  is integrally closed in  $K$ .
  - (ii) Prove or disprove: If  $a$  is a square-free element of  $D$ , that is,  $a$  is not divisible in  $D$  by the square of a prime element, then  $D[X]/(X^2 - a)$  is integrally closed in  $K[X]/(X^2 - a)$ .
  
4. Let  $p$  be a prime number,  $k$  a field,  $\beta$  a non-zero element of  $k$ , and  $K$  a splitting field over  $k$  for the polynomial  $f(X) = X^p - \beta$ .
  - (i) Prove that  $K$  contains a subfield  $K_1$  which is a splitting field over  $k$  for the polynomial  $X^p - 1$ , and that if  $\alpha$  is a root of  $f(X)$  in  $K$ , then  $K = K_1(\alpha)$ .
  - (ii) Show that either  $f(X)$  is irreducible in  $k[X]$ , or  $f(X)$  has a root in  $k$ .