Fall 2004 Algebra Preliminary Exam

Instructions: You have 4 hours. Solve three of the following four problems; indicate clearly which three you want graded.

Problem 1. Let $F = \mathbf{Z}/2\mathbf{Z}$ be the field of two elements. In this problem abelian groups in which every element has order dividing 2 will be regarded as vector spaces over F. If a is a nonzero rational number then we will write a' for its image in the group $\mathbf{Q}^*/(\mathbf{Q}^*)^2$.

Let $\mathbf{Q}$ be the rational field viewed as a subfield of the complex field $\mathbf{C}$. Let $a_1, \ldots, a_n \in \mathbf{Q}^*$ and let $K = \mathbf{Q}(\sqrt{a_1}, \ldots, \sqrt{a_n}) \subset \mathbf{C}$. Let A be the subgroup of $\mathbf{Q}^*/(\mathbf{Q}^*)^2$ generated by the a_i'.

a. Show that $K/\mathbf{Q}$ is Galois with abelian Galois group G, and that every element of G has order dividing 2. (*Hint:* adjoin the roots one by one.)

b. Consider the pairing $(,) : A \times G \longrightarrow F$ defined by the condition

$$(a', \sigma) = \begin{cases} 0 & \text{if } \sigma \text{ fixes } \sqrt{a} \\ 1 & \text{if } \sigma \text{ negates } \sqrt{a} \end{cases}$$

for all $a \in \mathbf{Q}^*$ with $a' \in A$. Show that this pairing is well-defined and bilinear.

c. Prove that the pairing $(,)$ is perfect; i.e. show that if $(a', \sigma) = 0$ for all $\sigma \in G$ then a is the identity of A and that if $(a, \sigma) = 0$ for all $a' \in A$ then σ is the identity in G. Conclude that A and G have the same dimension as vector spaces over F.

d. Show that for every r there is a Galois $K/\mathbf{Q}$ such that the Galois group of $K/\mathbf{Q}$ is $(\mathbf{Z}/2\mathbf{Z})^r$.

Problem 2. In this problem you may use without proof the fact that the alternating groups A_n are simple for $n \geq 5$.

a. Use Sylow's theorems to show that the symmetric group S_5 can be made to act transitively on a set of size six.

b. The symmetric group S_6 contains at least two conjugacy classes of subgroups that are isomorphic to S_5.

c. If X and Y are sets, then a bijection between them induces an isomorphism of the symmetric group of X with the symmetric group of Y.

d. There is a nontrivial outer automorphism of S_6—that is, an automorphism that is not the conjugation map of any element of S_6.

Problem 3. Suppose N is a linear transformation of an n-dimensional vector space over a field, and that $N^d = 0$ but $N^{d-1} \neq 0$.

a. What is the characteristic polynomial of $N + I$? (I being the identity map)

b. What is the minimal polynomial of $N + I$?

Of course, you must explain your answers.

c. Give an example where the minimal and characteristic polynomials are different.

Problem 4. In this problem you may assume that for any ring R and any module M over R, $R \otimes_R M \cong M$.

a. Let A, B be abelian groups, which are *not* assumed to be finitely generated. Suppose in $A \otimes_{\mathbf{Z}} B$ we have the relation $\sum a_i \otimes b_i = 0$. Prove that there are finitely generated subgroups $A' \subset A$ and $B' \subset B$ such that all $a_i \in A'$, all $b_i \in B'$, and $\sum a_i \otimes b_i = 0$ in $A' \otimes_{\mathbf{Z}} B'$. (You may use the description of the tensor product in terms of generators and relations.)

b. Consider the rational field $\mathbf{Q}$ as an abelian group. If $A' \subset \mathbf{Q}$ is a finitely generated nonzero subgroup, show that $A' \cong \mathbf{Z}$.

c. Consider $C = \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q}$. Show that if $\alpha \in C$ and $n\alpha = 0$ for some integer $n \neq 0$ then $\alpha = 0$.

d. Show that $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q} \otimes_{\mathbf{Q}} \mathbf{Q} \cong \mathbf{Q}$.