Preliminary Examination in Algebra

August 23, 2005, RLM 9.166, 1:00-5:00 p.m.

Do three of the following four problems.

(1.) Let G be a finite group, let $C \subseteq G$ be the center of G, and suppose that $K \subseteq G$ is a normal subgroup.

(i) Let p be a prime number and assume that p divides $|C|$ but p does *not* divide $|C \cap K|$. (Here $|C|$ is the cardinality of C and $|C \cap K|$ is the cardinality of $C \cap K$.) Show that p divides the index $[G : K]$.

(ii) Let $G' \subseteq G$ be the commutator subgroup of G, p a prime number, and assume that p divides $|C|$ but p does *not* divide $|C \cap G'|$. Show that G has a subgroup of index p.

(iii) Prove that

$$\mathcal{S}_p(K) = \{P \cap K : P \in \mathcal{S}_p(G)\},$$

where $\mathcal{S}_p(K)$ is the set of Sylow-p-subgroups of K and $\mathcal{S}_p(G)$ is the set of Sylow-p-subgroups of G.

(iv) Prove that $|\mathcal{S}_p(K)|$ divides $|\mathcal{S}_p(G)|$.

(2.) Let R be a principal ideal domain so that R^n is a free R-module of rank n.

(i) Prove that if $M \subseteq R^n$ is an R-submodule then M is free of rank at most n.

(ii) Let M be a finitely generated R-module. Show that there exist nonnegative integers $m \leq n$ and suitable maps so that

$$0 \to R^m \to R^n \to M \to 0$$

is an exact sequence of R-modules.

(iii) Let M be a finitely generated R-module. Prove that every R-submodule of M is again finitely generated.

(3.) Let V be a finite dimensional vector space over a field k, and $M : V \to M$ an endomorphism such that $M^2 = M$.

(i) Show that $\mathbf{1} - M$ is also idempotent, that is, $(\mathbf{1} - M)^2 = \mathbf{1} - M$, where $\mathbf{1} : V \to V$ is the identity map.

(ii) Write

$$\mathcal{R}(M) = \{M\mathbf{v} : \mathbf{v} \in V\} \quad \text{and} \quad \mathcal{N}(M) = \{\mathbf{v} \in V : M\mathbf{v} = \mathbf{0}\}$$

for the range and null space of M, respectively. Define $\mathcal{R}(\mathbf{1} - M)$ and $\mathcal{N}(\mathbf{1} - M)$ similarly. Prove that $V = \mathcal{R}(M) \oplus \mathcal{N}(M)$.

(iii) Show that an eigenvalue of M is either 0 or 1.

(iv) Show that V has a basis such that each basis vector is an eigenvector of M.

(4.) Let k be a field and let $f(x)$ be a separable, irreducible monic polynomial in $k[x]$ of degree p, where p is a prime number. In this problem we assume that K is a splitting field for $f(x)$ over k, $G = \operatorname{Aut}(K/k)$ is the Galois group of all automorphisms of K that fix k, $P \subseteq G$ is a Sylow-p-subgroup of G, $N \subseteq G$ is the normalizer of P in G, and $N = P$.

(i) Prove that $|G| = ps$ where s in an integer not divisible by p.

(ii) With $|G| = ps$, show that P has exactly s distinct conjugates in G.

(iii) With $|G| = ps$, show that there are exactly $s(p-1)$ distinct elements in G of order p.

(iv) Let α and β be distinct roots of $f(x)$ in K. Write $H_\alpha \subseteq G$ for the Galois group of K over $k(\alpha)$ and write $H_\beta \subseteq G$ for the Galois group of K over $k(\beta)$. Prove that $|H_\alpha| = s$.

(v) Show that

$$G = H_\alpha \cup \{\sigma \in G : \operatorname{ord}_G(\sigma) = p\},$$

and show that this is a disjoint union.

(vi) Prove that $H_\alpha = H_\beta$.

(vii) Prove that $K = k(\alpha)$ and that $P = G$.