#1. Let L/F be a finite extension of fields. Let $f(x0 \in F[x]$ be a polynomial.

a) Show that $F[x]/f(x)F[x] \otimes_F L \cong L[x]/f(x)L[x]$.

b) Suppose L, L' are both finite extensions of F. If L'/F is separable, use a) to show that $L \otimes L'$ is a direct sum of fields.

c) If L/F is Galois, show that $L \otimes_F L$ is a direct sum of copies of L.

d) If F has nonzero characteristic p and $L = F(a^{1/p})$, use nilpotence to show that $L \otimes_F L$ is not a direct sum of fields.

e) Suppose L/F is finite and $L \otimes_F L$ is a direct sum of copies of L. Show that L/F is Galois.

#2. Let F be an algebriacaly closed field and $M_n(F)$ the ring of n by n matrices over F. We say $A \in M_n(F)$ is diagonalizable if there is an invertible $P \in M_n(F)$ such that PAP^{-1} is a diagonal matrix. That is, all the off diagonal entries of PAP^{-1} are 0. $p(x) \in F[x]$ is called the minimum polynomial of A if $p(A) = 0$ and $p(x)$ is the monic polynomial of least degree with this property.

a) Show A is a diagonalizable matrix if and only if the minimum polynomial of A has distinct roots.

b) Now suppose $B \subset M_n(F)$ is a subalgebra which is simple as a ring. Show that there is a d dividing n and an invertible $P \in M_n(F)$ such that PBP^{-1} consists of all block diagonal matrices

$$\begin{pmatrix} A & 0 & 0 & \dots & 0 \\ 0 & A & 0 & \dots & 0 \\ . & & & & \\ . & & & & \\ . & & & & \end{pmatrix}$$

for all d by d matrices A.

c) Next suppose $B \subset M_n(F)$ has Jacobson radical 0. State a prove a version of b).

d) What does c) have to do with a)?

#3. Let G be a finite group and F a field. Let $F(G)$ be the field of fractions of the polynomial ring $F[x_g | g \in G]$. It is clear (and you do not have to prove) that there is an action of G on $F(G)$ with $g(x_{g'}) = x_{gg'}$. Let $F(G)^G$ be the fixed field. Let p be any prime. If L/K is a field extension, we say K' is an intermediate field of degree p if $K \subset K' \subset L$ and K'/K has degree p.

a) Show that $F(G)/F(G)^G$ is Galois with group G.

b) Suppose $L \supset K' \supset K$ are field extensions such that L/K is Galois with group G and G is a p group. If K'/K has degree greater than p, show that K'/K has an intermediate field of degree p.

Assume L/K is separable of degree p^2, and K' is an intermediate field of degree p. Write $K' = K(\theta)$ and $L = K(\alpha)$. Let $g(x), f(x) \in K[x]$ be the monic minimal polynomials of θ, respectively α over K. Let K'' be the field generated by all the roots of $g(x)$. Let G be the Galois group of K''/K and H that of K''/K'. Choose g_i such that G is the disjoint union of the cosets g_iH.

c) Show that $f(x)$ has an irreducible factor, $h(x)$, of degree p in $K'[x]$. Using the $g_i(h(x)$, show that $f(x)$ is the product of p factors of degree p over $K''[x]$.

d) Suppose L' is the extension generated by all the roots of $f(x)$. Show that L'/K has degree no more than $(p!)^{p+1}$.

e) Show that there is a field extencion L/K of degree p^2 with no intermediate field of degree p.