Preliminary Examination in Algebra

January 10, 2006, RLM 9.166, 1:00-5:00 p.m.

Do five of the following six problems.

(1.) Let G be the group given by generators and relations as

$$\langle x, y, z : xy^{-1}[x, y] = yz^{-1}[y, z] = zx^{-1}[z, x] = 1_G \rangle,$$

where $[a, b] = aba^{-1}b^{-1}$ denotes the commutator of elements a and b. Prove that G is an infinite group.

(2.) Let E/K be a Galois extension of fields of degree $p^2 q$, where p and q are primes, $p > q$, and q does not divide $p^2 - 1$.

(i) Prove that there exist unique intermediate fields L and M such that

$$K \subseteq L \subseteq E, \quad K \subseteq M \subseteq E,$$

$[L : K] = p^2$, and $[M : K] = q$.

(ii) Prove that the fields L and M are each Galois over K.

(iii) Prove that the Galois group $\mathrm{Aut}(E/K)$ is abelian

(3.) Let $\varphi : \mathbb{Z}^N \to \mathbb{Z}^N$ be a homomorphism of $\mathbb{Z}$-modules.

(i) Let $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_N$ be a basis for $\mathbb{Z}^N$ as a $\mathbb{Z}$-module and let A be the $N \times N$ matrix with entries in $\mathbb{Z}$ determined by φ and this basis. Prove that $|\det A|$ is independent of the choice of basis and hence $|\det \varphi|$ is well defined.

(ii) Let

$$\mathcal{X} = \mathbb{Z}^N / \{\varphi(\mathbf{z}) : \mathbf{z} \in \mathbb{Z}^N\}$$

denote the cokernel of φ. Prove that $\mathcal{X}$ is finite if and only if $|\det \varphi| \neq 0$.

(iii) Assume that $|\det \varphi| \neq 0$ and show that the cardinality of $\mathcal{X}$ is $|\det \varphi|$.

(4.) Let $R = \mathbb{Z}[\sqrt{2}]$ and let $N : R \to \{0, 1, 2, \ldots\}$ be defined by $N(a + b\sqrt{2}) = |a^2 - 2b^2|$.

(i) Prove that N is a norm on the ring R and R is a Euclidean domain with respect to this norm.

(ii) Prove that the element $1 + \sqrt{2}$ in R is a unit of infinite order.

(iii) Explain why the identities

$$\begin{aligned} 7 &= (5 + 3\sqrt{2})(5 - 3\sqrt{2}) \\ &= (27 + 19\sqrt{2})(27 - 19\sqrt{2}) \\ &= (75 + 53\sqrt{2})(75 - 53\sqrt{2}) \end{aligned}$$

do not violate unique factorization in R.

(iv) Determine a factorization of 7 into prime elements of R.

(5.) Let $K = \mathbb{F}_{1024}$ be the finite field with $1024 = 2^9$ elements, and let L/K be a field extension of degree 2.

(i) Prove that there is a unique automorphism $\sigma : L \to L$ which fixes K and has order 2.

(ii) Determine the number of elements x in the multiplicative group $L^\times$ such that $\sigma(x) = x^{-1}$.

(6.) Let V and W be finite dimensional vector spaes over the field k. Let V^* be the dual vector space of all linear maps from V into k. Define a map

$$\psi : V^* \otimes W \to \mathrm{Hom}(V, W)$$

by

$$\psi(\mathbf{v}^* \otimes \mathbf{w})(\mathbf{v}) = \mathbf{v}^*(\mathbf{v})\mathbf{w}.$$

(i) Prove that ψ is well defined and a homomorphism.

(ii) Prove that ψ is injective.

(iii) Prove that ψ is surjective.