PRELIMINARY EXAMINATION IN ALGEBRA

August 1999 4 Hours (1:00 - 5:00)

Directions: Do three problems out of four. Indicate clearly which three problems you have done. In general, it is permissible to use earlier parts of a problem to do a later part even if you have not demonstrated the earlier part.

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1. Assume that V is a commutative integral domain with quotient field F and that V has the property that $\forall\ \alpha \in F$ with $\alpha \neq 0$, either $\alpha \in V$ or $\alpha^{-1} \in V$. [Such a V is called a valuation domain.]
   a) Let I, J be ideals of V. Show that either $I \subseteq J$ or $J \subseteq I$. [For future reference, note that this implies that V has a unique maximal ideal.]
   b) Suppose I is a finitely generated ideal of V. Show that I is a principal ideal. [Hint: Consider ideals with two element generating sets.]
   c) Let $0 \neq \alpha \in F$ and suppose there exists $a_0, \ldots, a_{n-1} \in V$ such that $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 \in V$. Show that $\alpha \in V$.
   d) Let W be an integral domain with $V \subseteq W \subseteq F$. Show that W is also a valuation domain.
   e) Let $V \subseteq W \subseteq F$ as in (d) and let I be a proper ideal of W. Show that $I \subset V$.
   f) Let M be the unique maximal ideal of W. Show that $W = \{\frac{a}{s} \mid a \in V,\, s \in V - M\}$.
2. Let G be a group of order 5075 ($= 5^2 \cdot 7 \cdot 29$). Let P be a Sylow 5-subgroup of G, let Q be a Sylow 29-subgroup of G, and let R be a Sylow 7-subgroup of G.
   a) Show that P is a normal subgroup of G.
   b) Show that G has a normal subgroup H of order $5^2 \cdot 29$. [Hint: Look at G/P.]
   c) Show that Q is a normal subgroup of G. [Hint: Relate Q to H.]
   d) Show that G has a subgroup K of order $5^2 \cdot 7$.
   e) Show that K is normal if and only if G is abelian.
3. Let $F \subseteq F(\alpha) \subseteq E$ with E/F a finite Galois extension. Let G be the Galois group of E/F and let H be the Galois group of $E/F(\alpha)$.
   a) Suppose σ and τ are both in G and let σH and τH be the corresponding left cosets of H. Show that $\sigma H = \tau H$ if and only if $\sigma(\alpha) = \tau(\alpha)$.

   Now let $\sigma_1, \ldots, \sigma_n$ be a complete set of left coset representatives for H in G.
   b) Let τ be an arbitrary element of G. Show that $\tau\sigma_1, \ldots, \tau\sigma_n$ is a complete set of left coset representatives for H in G.
   c) Let $g(X) = (X - \sigma_1(\alpha))(X - \sigma_2(\alpha)) \cdots (X - \sigma_n(\alpha))$. Show that $g(X) \in F[X]$.
   d) Show that $g(X)$, as defined in part (c), is the minimal polynomial for α over F.
4. Let F be a field and $R = \mathbb{M}_n(F)$ the ring of $n \times n$ matrices over F. Let $V = F^n$ be the space of column vectors of length n, viewed as a left R-module in the usual way.
   a) Show that the only submodules of V are $\{0\}$ and V.
   b) Show that R is a direct sum of left ideals L_i, each of which is isomorphic to V as an R-module.
   c) Let W be a left R-module and let $x \in W$. Let L be one of the ideals L_i in (b). Show that if $Lx \neq \{0\}$, then Lx is a submodule of W which is isomorphic to V.
   d) Let W be as in (c) and further assume that the dimension of W as a vector space over F is finite (viewing F as the subring of R consisting of scalar matrices). Prove that W is isomorphic as an R-module to a finite direct sum of copies of V.