Preliminary Examination in Analysis

August 16, 2004, RLM 9.166, 1:00-5:00 p.m.

Part 1: Real Analysis

(1.) Let Ω_1 and Ω_2 be Borel subsets of $\mathbb{R}^n$ and $Y : \Omega_1 \to \Omega_2$ a measure preserving transformation. That is,

$$|Y^{-1}(A)|_{\mathcal{L}^n} = |A|_{\mathcal{L}^n}$$

for all measurable subsets $A \subseteq \Omega_2$, where $|\ |_{\mathcal{L}^n}$ is n-dimensional Lebesgue measure. Show that the map $f(\mathbf{y}) \to f\big(Y(\mathbf{x})\big)$ is an isometry from $L^p(\Omega_2)$ to $L^p(\Omega_1)$ for all $0 < p \le \infty$.

(2.) Define two measures μ_1 and μ_2 on the Borel sets of $\mathbb{R}^2$ as follows. For a Borel set $B \subseteq \mathbb{R}^n$,

$$\mu_1(B) = \big|B \cap \{\mathbf{x} \in \mathbb{R}^2 : x_2 = 0\}\big|_{\mathcal{L}^1} + \int_B e^{-|\mathbf{x}|^2}\ \mathrm{d}\mathbf{x},$$

and

$$\mu_2(B) = \big|B \cap \{\mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1\}\big|_{\mathcal{L}^2}.$$

Here $|\ |_{\mathcal{L}^1}$ is one dimensional Lebesgue measure and $|\ |_{\mathcal{L}^2}$ is two dimensional Lebesgue measure. Determine the Radon-Nikodym decomposition of μ_1 with respect to μ_2.

(3.) Let $\mathcal{A} \subseteq L^2[0, 2\pi]$ be the set of all functions $f(x)$ such that

$$f(x) = \lim_{K\to\infty} \sum_{k=-K}^{K} a_k e^{ikx} \quad \text{in } L^2\text{-norm, with} \quad |a_k| \le (1 + |k|)^{-1}.$$

Show that $\mathcal{A}$ is compact in $L^2[0, 2\pi]$.

(4.) Let $g : \mathbb{R} \to \mathbb{C}$ be a bounded, measureable, periodic function with period 1, and f a function in $L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) g(nx)\ \mathrm{d}x = \Big(\int_0^1 g(s)\ \mathrm{d}s\Big)\Big(\int_{\mathbb{R}} f(t)\ \mathrm{d}t\Big).$$

Part 2: Complex Analysis

(5.) Let $f(z)$ be an entire function such that $|f(z)| \to \infty$ as $|z| \to \infty$. Prove that f must be a polynomial.

(6.) Suppose that $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and for each $n = 1, 2, \ldots$ let $f_n : \Delta \to \mathbb{C}$ be an analytic function. Assume that:

(i) The inequality $|f_n(z)| \leq (1 - |z|)^{-1/2}$ holds at each point z in Δ.

(ii) The limit

$$\lim_{n\to\infty} f_n(z) = F(z)$$

exists at each point z in Δ.

Prove that the resulting function $F : \Delta \to \mathbb{C}$ is analytic.

(7.) Let $\alpha \neq 0$ be in $\mathbb{C}$.

(i) Determine the Laurent series expansion for the function $z \to \exp\{\frac{1}{2}\alpha(z - z^{-1})\}$ in the annulus $A = \{z \in \mathbb{C} : 0 < |z| < \infty\}$.

(ii) Use this information to evaluate the integral

$$\frac{1}{2\pi i} \int_\gamma z^{-1} \exp\{\tfrac{1}{2}\alpha(z - z^{-1})\}\ \mathrm{d}z,$$

where $\gamma(\theta) = e^{2\pi i\theta}$ and $0 \leq \theta \leq 1$.

(iii) Use the formula in (ii) to conclude that

$$\int_0^1 \cos\{\alpha \sin(2\pi\theta)\}\ \mathrm{d}\theta = \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha/2)^{2n}}{(n!)^2}.$$

(8.) Let $f(z)$ be analytic in the open disk $\{z \in \mathbb{C} : |z| < 1\}$, continuous in the closed disk $\{z \in \mathbb{C} : |z| \leq 1\}$, and satisfy $f(0) = 0$. Write

$$f(z) = u(z) + iv(z),$$

where $u(z)$ and $v(z)$ are the real and imaginary parts of $F(z)$, respectively. Prove that

$$\max\{v(z) : |z| \leq r\} \leq \frac{2r}{1 - r^2} \max\{u(w) : |w| \leq 1\},$$

where $0 < r < 1$.