PRELIMINARY EXAMINATION IN ANALYSIS — JANUARY 2003

Directions: Time allowed is four hours. Work on as many problems as possible, but do at least three problems from each section.

Real Analysis

Here m denotes the Lebesgue measure.

1. Let $\{E_k\}$ be a sequence of Lebesgue measurable sets in $\mathbb{R}^n$ such that $A = \sum_{k=1}^{\infty} m(E_k) < \infty$. Let F_j be the subset of $\mathbb{R}^n$ such that each point occurs in precisely j of the subsets E_k. Then show that $A = \sum_{j=1}^{\infty} jm(F_j)$.

2. Let $F = (f_1, \cdots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map, i.e., each $f_i \in \mathbb{R}[x_1, \cdots, x_n]$ for $i = 1, \cdots, n$. Show that for every set $E \subset \mathbb{R}^n$ of Lebesgue measure zero, then $F(E)$ has measure zero.

3. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ define convolution by
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy \ .$$
Prove that for all $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$
$$f * g \text{ is continuous with } \lim_{|x|\to\infty} (f*g)(x) = 0 \ .$$

4. Let f be a real-valued Lebesgue measurable function on $\mathbb{R}^n$ and define the distribution function $\lambda(\alpha) = m\{x \in \mathbb{R}^n : |f(x)| > \alpha\}$
   (a) For $f \in L^1(\mathbb{R}^n)$ show that
   $$\int_{\mathbb{R}^n} |f|\,dx = \int_0^{\infty} \lambda(\alpha)\,d\alpha \ .$$
   (b) Suppose that $f \in L^1_W(\mathbb{R}^n)$, i.e., $\sup_{\alpha>0}[\alpha\lambda(\alpha)] = M < \infty$. Show that for $0 < p < 1$ and E a subset of finite measure
   $$\int_E |f|^p\,dx < \infty \ .$$
   (c) Extend the argument of (b) to show the estimate
   $$\int_E |f|^p\,dx \le c(n,p)\big[m(E)\big]^{1-p} M^p$$
   for a suitable constant $c(n,p)$.

Complex Analysis

Here $\mathbb{C}$ denotes the complex plane.

1. Use the method of contour integration to compute the integral for $t \in \mathbb{R}$

$$\int_0^\infty \frac{\cos(xt)}{1+x^2}\,dx \ .$$

2. Suppose that f is an entire function and that $|f(z)| \leq A|z|^m$ for $|z| > 1$ (m is a positive integer). Show that f is a polynomial of degree at most m.

3. Let H denote the upper half plane $\{z : \operatorname{Im} z > 0\}$. Suppose $f : \bar{H} \to \mathbb{C}$ is bounded and continuous with f analytic on H. Assume that $|f(x)| \leq M < \infty$ for all real x. Prove that $|f(z)| \leq M$ for all $z \in H$.

4. Let F be a finite set in $\mathbb{C}$ and define $D = \mathbb{C} \sim F$. Let f be analytic in D with the property that

$$\lim_{|z|\to\infty} |f(z)| = \infty \quad \text{and} \quad \lim_{z\to z_\alpha} |f(z)| = \infty \quad \text{for each} \quad z_\alpha \in F \ .$$

   Then show that $f(D) = \mathbb{C}$.