PRELIMINARY EXAMINATION IN ANALYSIS
January 12, 2004, 1:00 p.m., RLM 9.166

There are 8 problems: 4 Real Analysis Problems and 4 Complex Analysis Problems. Time allowed is 4 hours.

Real Analysis

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function in $L^1_{\text{weak}}(\mathbb{R}^n)$, i.e.,

$$\theta = \sup_{\lambda > 0} \lambda \big| \{x : |f(x)| > \lambda\} \big| < \infty$$

   Prove that for every $\delta \in (0, 1)$, and for every measurable set $E \subset \mathbb{R}^n$, $|E| < \infty$

$$\int_E |f|^\delta \, dx \leq C |E|^{1-\delta} \theta^\delta \ ,$$

   with an appropriate constant C independent of f and E.

2. Construct a (nonmeasurable) function $f : \mathbb{R} \to \mathbb{R}$ with the following property:
   For any $g : \mathbb{R} \to \mathbb{R}$ such that

$$|g(x) - f(x)| < 1 \quad \forall\, x \in \mathbb{R}$$

   g is *not* measurable.

3. Prove that for any $f \in L^1(\mathbb{R})$

$$\lim_{\varepsilon \to 0+} \frac{1}{\sqrt{\pi\varepsilon}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{\varepsilon}} f(y) \, dy = f(x)$$

   for almost all $x \in \mathbb{R}$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous with $f' \in L^1(-\infty, \infty)$. Prove that for every $h \in \mathbb{R}$

$$\int_{-\infty}^{\infty} |f(x+h) - f(x)| \, dx \leq C|h|$$

   for a suitable constant C independent of h.

Complex Analysis

1. Let f be as an entire analytic function such that $|f(x+iy)| \le e^{|y|}$ for $x, y \in \mathbb{R}$. Show that for any z that is not an integer multiple of π,

$$\frac{d}{dz}\left(\frac{f(z)}{\sin z}\right) = -\sum_{n=-\infty}^{\infty} \frac{(-1)^n f(n\pi)}{(z-n\pi)^2}$$

where the right-hand side converges uniformly on every compact subset in $\mathbb{C}$ that does not contain any integer multiple of π.

2. For γ the positively oriented circle $|z| = 2$, compute the integrals

$$\int_\gamma \frac{z^3 dz}{z^5+2}\ , \qquad \int_\gamma (e^{\pi z}+1)^{-1}\, dz\ , \qquad \int_\gamma \sqrt{z^2-1}\, dz\ ,$$

when the branch of the square root is chosen such that $\sqrt{3} > 0$.

3. Show that a positive harmonic function on $\mathbb{R}^2$ is constant.

4. Suppose f is analytic in an open rectangle $R \subset \mathbb{C}$ with corners ± 1, $i \pm 1$. Prove that there is a sequence $\{z_n\}$ in R, approaching the boundary of R, such that the sequence $\{f(z_n)\}$ is bounded.