

ANALYSIS PRELIM – AUGUST 2001

Time allowed is four hours. Work on as many problems as possible, but do at least three problems from each section.

**Real Analysis**

Here  $m$  denotes Lebesgue measure on  $\mathbb{R}^n$ .

1. Find a sequence of real-valued nonnegative functions  $\{f_k\}$  on  $[0,1]$  so that

$$(a) \limsup f_k = +\infty \forall x, \quad (b) \int_{[0,1]} f_k dx \rightarrow 0.$$

Adapt this argument to the case where the domain is  $\mathbb{R}^n$ .

2. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a Lebesgue measurable function such that

$$m\{x : |f(x)| > \lambda\} \leq C\lambda^{-2}, \quad \lambda > 0.$$

Prove that there is a constant  $C_1$  such that for any Borel set  $E \subset \mathbb{R}^n$  of finite and positive measure

$$\int_E |f(x)| dx \leq C_1 \sqrt{m(E)}.$$

3. For  $g \in L^1(\mathbb{R})$  and any  $f \in C^1(\mathbb{R})$ , suppose that

$$g * f'(x) = f(x+h) - f(x-h)$$

Show that  $g \equiv \chi_{[-h,h]}$ .

4. Let  $B_{m,1}$  denote the ball of radius one centered at the origin in  $\mathbb{R}^m$ .

- a) Show that there exists a function  $f : \mathbb{R} \rightarrow [0, 1]$  such that

$$m\{B_{n+1,1}\} = m\{B_{n,1}\} \int [f(t)]^n dt$$

- b) Show that  $\int [f(t)]^n dt \rightarrow 0$  as  $n \rightarrow \infty$ .

- c) Show that for any finite positive number  $A$ ,  $A^n m\{B_{n,1}\} \rightarrow 0$  as  $n \rightarrow \infty$ .

5. Let  $\{f_n\}$  be a sequence of real-valued functions in  $L^2$  such that  $\|f_n\|_{L^2} = 1$ , and for  $n < m$

$$\left| \int f_n f_m dx \right| \leq 2^{-m} \rightarrow 0, \quad m \rightarrow \infty$$

Let  $\{e_n\}$  be a Gram-Schmidt orthogonalization for this sequence with  $\|e_n\| = 1$ .

a) Show that

$$\|f_m - e_m\|^2 \leq 2^{-m}.$$

b) For any  $g \in L^2$ , show that as  $m \rightarrow \infty$ ,  $\int g f_m dx \rightarrow 0$ .

6. Consider a real-valued nonnegative function  $g \in L^1[0, 1]$ . Suppose that  $\int g f dx < A$  whenever  $\int e^f dx \leq 1$ .

a) What can you say about the measure of the set  $\{g > \lambda\}$  for large  $\lambda$ ? (try  $f = c\chi_E$ )

b) Can you say that  $g \in L^2$ ? Why or why not?

### Complex Analysis

Here  $\mathbb{C}$  denotes the complex plane,  $\Omega$  denotes a region in  $\mathbb{C}$ ,  $dx dy$  denotes Lebesgue measure on  $\mathbb{C}$ .

1. Suppose that  $\{f_n\}$  is a sequence of analytic functions on a region  $\Omega$ , and  $F$  a continuous function on  $\Omega$  such that for any disk  $D \subset \Omega$ ,

$$\lim_{n \rightarrow \infty} \int_D |f_n(z) - F(z)| dx dy = 0.$$

Show that  $F$  is analytic on  $\Omega$ , and that  $f_n \rightarrow F$  and  $f'_n \rightarrow F'$  uniformly on compact subsets of  $\Omega$ .

2. Let  $p(z)$  be a polynomial of degree  $N \geq 1$  and define

$$L = \{z \in \mathbb{C} : |p(z)| = 1\}.$$

Prove that the open set  $\mathbb{C} - L$  has at most  $N + 1$  components.

3. Suppose  $f$  is a complex-valued function defined on  $[0, \infty)$  that belongs to  $L^p([0, \infty))$  for some  $p$ ,  $1 < p < \infty$ . Define  $G$  on the right half-plane by

$$G(z) = \int_0^\infty f(t) e^{-tz} dt.$$

Prove that  $G(z)$  is analytic for  $\Re z > 0$ . What additional assumption on  $f$  would ensure that  $G$  is entire?

4. Use a contour integral to compute the value of one of the following integrals:

a) the improper Riemann integral

$$\int_0^{\infty} e^{ix^2} dx$$

b) the Fourier transform of a Gaussian function on  $\mathbb{R}$

$$\int_{-\infty}^{\infty} e^{ixy} e^{-x^2} dx.$$

5. Consider a compact set  $K \subset \mathbb{R}$  with positive Lebesgue measure. On  $\mathbb{C} - K$  define the function

$$g(z) = \int_K \frac{1}{t - z} dt.$$

a) Prove that  $g$  is analytic on  $\mathbb{C} - K$ ;

b) Prove that  $g$  cannot be extended by analytic continuation to an entire function;

c) Show that  $\lim_{z \rightarrow \infty} [zf(z)]$  exists and determine its value.

6. a) Construct a one-to-one conformal mapping of the upper half-plane

$H = \{z \in \mathbb{C} : \Im z > 0\}$  onto the angle region with an interval removed

$$A = \{z \in \mathbb{C} : z \neq 0, \arg z \in (-\pi/4, \pi/4)\} - \{z : \Im z = 0, 0 \leq \Re z \leq 1\}.$$

b) Describe all the one-to-one conformal mappings of  $H$  onto  $A$ .

