Preliminary Examination in Analysis
January 8, 2001, 1:00-5:00 p.m.

Part 1: Real Analysis

1.) Let $f_n : [0,1] \to \mathbb{R}$ be Lebesgue measurable with

$$(*) \qquad \int_0^1 |f_n(t)|^3 \, dm(t) \le 1 \quad \text{for all} \quad n \in \mathbb{N} \; .$$

a) Show that for all $\varepsilon > 0$ there exists $\delta > 0$ so that if $E \subseteq [0,1]$ is Lebesgue measurable with $m(E) < \delta$ then

$$\int_E |f_n(t)| \, dm(t) < \varepsilon \quad \text{for all} \quad n \in \mathbb{N} \; .$$

b) Show by example that the conclusion of a) is false if $(*)$ is replaced by

$$\int_0^1 |f_n(t)| \, dm(t) \le 1 \quad \text{for all} \quad n \in \mathbb{N} \; .$$

2.) True or False (justify your answers)

a) If $A \subseteq \mathbb{R}$ is Lebesgue measurable there exists a Borel set $B \subseteq \mathbb{R}$ with $m((A \setminus B) \cup (B \setminus A)) = 0$.

b) If $f : [0,1] \to \mathbb{R}$ is Lebesgue measurable then there exists a Borel measurable function $g : [0,1] \to \mathbb{R}$ with

$$m\{t : f(t) \ne g(t)\} = 0 \; .$$

3.) Let $f_n : \mathbb{R} \to [0, \infty)$ be nondecreasing for each $n \in \mathbb{N}$. Assume that for all $x \in \mathbb{R}$

$$f(x) \equiv \sum_{n=1}^{\infty} f_n(x) < \infty \; .$$

Prove that

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \quad \text{a.e.}$$

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Part 2: Complex Analysis

4.) Let $f(z)$ be analytic on $\mathbb{C} \setminus \{1\}$ and have a simple pole at $z = 1$ with residue λ. Prove that for every $T > 0$,

$$\lim_{n\to\infty} T^n \left| (-1)^n \frac{f^{(n)}(2)}{n!} - \lambda \right| = 0 \ .$$

5.) Let $f(z)$ be analytic on an open set that contains

$$\overline{\Delta} = \{\omega \in \mathbb{C} : |\omega| \leq 1\} \ .$$

Assume that $|f(z)| = 1$ if z is on the boundary of $\overline{\Delta}$ and that f has N zeros in the interior of $\overline{\Delta}$. Prove that $f'(z)$ has $N - 1$ zeros in the interior of $\overline{\Delta}$. Here all zeros are counted with multiplicity.

6.) Suppose that $f(z)$ is an entire function such that

$$|f(z)| \leq B\, e^{A|z|} \ , \qquad z \in \mathbb{C} \ ,$$

for some positive constants A and B. Let $\omega_1, \omega_2, \ldots$ be the zeros of f listed with appropriate multiplicity. Prove that

$$\sum_{n=1}^{\infty} (1 + |\omega_n|)^{-\alpha} < \infty$$

for all $\alpha > 1$.