ANALYSIS PRELIMINARY EXAMINATION — August 2003

Directions: Work at least 4 problems from each section. Time allowed is 4 hours.

Real Analysis

1. Let $f : [0,1] \to [0,1]$ be measurable, one-to-one and onto, and satisfies the following condition:
   - $\forall\, N \subset [0,1]\ , \quad m(N) = 0 \Longrightarrow m(f(N)) = 0$

   Prove that $f^{-1} : [0,1] \to [0,1]$ is measurable. Here m denotes Lebesgue measure.

2. Assume that $F : [0,1] \to \mathbb{R}$ is absolutely continuous and
$$\int_0^1 F(x)\ \mathrm{d}x = 0\ .$$
   Prove that
$$\sup_{0\le y\le 1} \left| \int_0^1 (x-y)F'(x)\ \mathrm{d}x \right| \le \sup_{0\le x\le 1} |F(x)|\ .$$

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable, $E =$ support of f and
$$(*) \qquad \int_E e^{|f(x)|}\ \mathrm{d}x = 1\ .$$
   Prove that $f \in L^p(\mathbb{R}^n)\ \forall\, p \in (0,\infty)$ and
$$\|f\|_{L^p} \le C\,p\ , \qquad p \in (0,\infty)$$
   where the constant C does not depend on f or p. Give an example of f satisfying $(*)$ such that $f \notin L^\infty$.

4. Let $p : \mathbb{R}^n \to [2,4]$ be measurable, $f_n : \mathbb{R}^n \to \mathbb{R}$, $n = 1,2,3,\ldots$ be a sequence of measurable functions such that
   - $\displaystyle\int_{\mathbb{R}^n} \left|f_n(x)\right|^{p(x)}\ \mathrm{d}x < \infty$
   - $\displaystyle\lim_{m,n\to\infty} \int_{\mathbb{R}^n} \left|f_n(x) - f_m(x)\right|^{p(x)}\ \mathrm{d}x = 0$

   Prove there is a measurable function
$$f : \mathbb{R}^n \to \mathbb{R}$$
   such that
   - $\displaystyle\int_{\mathbb{R}^n} |f(x)|^{p(x)}\ \mathrm{d}x < \infty$
   - $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}^n} |f_n(x) - f(x)|^{p(x)}\ \mathrm{d}x = 0$

   Hint: Start with $p(x)$ being a simple measurable function; then adjust your argument.

5. Let $f \in L^2(\mathbb{R})$ and define

$$\psi(x) = \frac{1}{\sqrt{|x|}} \frac{\sin x}{x} \ , \quad (\psi * f)(x) = \int_{\mathbb{R}} \psi(x-y) f(y) \, dy$$

Show that $\psi * f \in L^2(\mathbb{R})$.
Determine whether $\psi * f$ is a continuous function on $\mathbb{R}$.
For $\varepsilon > 0$, define $\psi_\varepsilon(x) = \varepsilon^{-1}\psi(x/\varepsilon)$.
Show that $\psi_\varepsilon * f$ has a limit in $L^2(\mathbb{R})$ as $\varepsilon \to 0$ and determine that limit.
What can be said about the pointwise limit

$$\lim_{\varepsilon \to 0} (\psi_\varepsilon * f)(x)$$

Explain.

Complex Analysis

1. Let $\mathcal{D} \subseteq \mathbb{C}$ be a nonempty, connected, open set. Suppose that $f_1(z), f_2(z), \ldots$ are analytic and not zero on $\mathcal{D}$. Also assume that

$$\lim_{n \to \infty} f_n(z) = F(z)$$

uniformly on compact subsets of $\mathcal{D}$. Prove that either $F(z) = 0$ for all z in $\mathcal{D}$ or $F(z)$ has no zeros on $\mathcal{D}$.

2. Let $\mathcal{D} \subseteq \mathbb{C}$ be a nonempty, connected, open set. Suppose that $f : \Delta \to \mathcal{D}$ and $g : \Delta \to \mathcal{D}$ are analytic and bijective. Here $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ Also assume that

$$f(0) = g(0), \quad f'(0) > 0, \quad \text{and} \quad g'(0) > 0 \ .$$

Prove that $f(z) = g(z)$ for all z in Δ.

3. Suppose that $f_1(z), f_2(z), \ldots, f_N(z)$ are continuous complex valued functions on the closed unit disk $\overline{\Delta} = \{z \in \mathbb{C} : |z| \leq 1\}$ and analytic on the interior Δ. Prove that the function

$$g(z) = |f_1(z)| + |f_2(z)| + \cdots + |f_N(z)|$$

takes on its maximum value on the boundary of $\overline{\Delta}$. That is, prove that

$$\sup\{g(z) : |z| \leq 1\} = \sup\{g(z) : |z| = 1\} \ .$$

4. Show that if $r > 1$ then the map $z \mapsto e^{z-r}$ has exactly one fixed point in the half-plane $\mathrm{Re}(z) < 1$.

5. Evaluate the integral

$$\int_0^\infty \frac{\cos(\pi x)}{4x^2 - 1} \, \mathrm{d}x \ .$$