PRELIMINARY EXAMINATION IN ANALYSIS
August 22, 2005, 1:00 pm, RLM 9.166

There are 10 problems: 5 Real Analysis Problems and 5 Complex Analysis Problems. Time allowed is 4 hours.

Real Analysis

1. Let $f \in L^2(\mathbb{R}^2)$, and define

$$g_n(x) = \big(\phi_n * f\big)(x) = \int_{\mathbb{R}^2} \phi_n(x-y) f(y) dy \,, \qquad \phi_n(x) = n^4 x_1 x_2 e^{-n|x|} \,,$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, where $|x| = \sqrt{x_1^2 + x_2^2}$. Show that $\{g_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^2)$, and determine its limit.

2. Define $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$ for $r > 0$, and

$$W_\varepsilon(x) = \frac{\partial^2}{\partial x_1^2} \sqrt{x_1^2 + \varepsilon} \,, \qquad x \in \mathbb{R}^2 \,, \quad \varepsilon > 0.$$

a) Show that W_ε converges weakly as $\varepsilon \to 0$, in the dual of $C\big(\overline{B_1}\big)$, to a measure μ.
b) What is $\mu(B_r)$?

3. Let χ_{A_k} be the characteristic functions for a sequence of measurable sets A_k. Show that if $\chi_{A_k} \to f$ in L^1, then $f = \chi_A$ a.e. for some set A.

4. Let $f \in L^1(\mathbb{R})$, and assume that for every $x \in \mathbb{R}$, there exists a sequence of intervals $I_{x,k}$ containing x, such that

$$\big|I_{x,k}\big| \to 0 \,, \qquad \frac{1}{\big|I_{x,k}\big|} \int_{I_{x,k}} f(y) dy \to 0 \,,$$

as $k \to \infty$. Show that $f = 0$ a.e.

5. Suppose that $f, f_1, f_2, \ldots$ are Lebesgue measurable functions in $L^4(\mathbb{R}^n)$, such that $f_k(x) \to f(x)$ a.e. and $\|f_k\|_{L^4} \to \|f\|_{L^4}$. Show that $f_k \to f$ in $L^4(\mathbb{R}^n)$.

Complex Analysis

6. Use the method of contour integration to evaluate

$$\int_0^\pi \frac{d\vartheta}{1+\sin^2(\vartheta)}\,.$$

7. (a) Let f_0 and f_1 be analytic in an open set $\Omega \subset \mathbb{C}$, and let γ be a regular Jordan curve in Ω. Define $f_s = (1-s)f_0 + sf_1$ for every real number $0 < s < 1$. Show that if none of the functions f_s vanishes on γ, then f_0 and f_1 have the same number of zeros in the region enclosed by γ.
(b) Use the above to show that $z^8 + \lambda z^7 + 1 = 0$ with $\lambda \in \mathbb{R}$ has exactly 4 roots with positive real parts.

8. Find all entire functions f satisfying $|f(z)| \le |f(2z)|^{1/2}$ for all $z \in \mathbb{C}$.

9. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Show that, among the analytic functions $f : D \to D$ satisfying $f(0) = 0$, there are some that maximize $|f''(0)|$, and find such a function.

10. Let f be a function of two complex variables, that is continuous on the closure of D^2, where $D = \{z \in \mathbb{C} : |z| < 1\}$. Assume that $z \mapsto f(z, z_2)$ and $z \mapsto f(z_1, z)$ are analytic on D, for every $z_1, z_2 \in D$. Show that f admits a power series expansion

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} c_{m,n} z_1^m z_2^n\,,$$

that converges uniformly on compact subsets of D^2.