# Preliminary Examination in Analysis
## January 10, 2005 — RLM 9.166 — 1:00–5:00 pm

### Part 1: Real Analysis

1. (a) Let $f_j : \mathbb{R}^N \to \mathbb{R}$ be a measurable function for each $j = 1, 2, \dots$, and suppose that

$$\int_{\mathbb{R}^N} |f_j(x)| \, \mathrm{d}x \leq 2^{-j} \ .$$

Show that $f_j(x) \to 0$ a.e. in $\mathbb{R}^N$.

(b) Is this result true if $\int_{\mathbb{R}^n} |f_j| \, \mathrm{d}x \leq \frac{1}{j}$?

2. Let $\{u_j\}$ be a sequence of measurable sets contained in the unit cube $Q_1$ in $\mathbb{R}^n$. Show that if $\sum m(u_j) = +\infty$, then

$$\sum_{i \neq j} m(u_i \cap u_j) = +\infty \ .$$

Here $m$ denotes Lebesgue measure in $\mathbb{R}^n$.

3. Let $f$ be in $L^1(\mathbb{R}, m)$ where $m$ is Lebesgue measure and then define $F : \mathbb{R} \to [0, \infty]$ by

$$F(x) = \sup_I \frac{1}{m(I)} \int_I |f(x)| \, \mathrm{d}m(x) \ ,$$

where the supremum is taken over all bounded open intervals $I \subseteq \mathbb{R}$ containing $x$.

(i) Prove that if $I_1, I_2, \dots, I_N$ is a finite collection of bounded open intervals in $\mathbb{R}$ then there exists a disjoint subcollection $J_1, J_2, \dots, J_M$ such that

$$m\Big( \bigcup_{n=1}^{N} I_n \Big) \leq 2 \sum_{m=1}^{M} m(J_m) \ .$$

(ii) Show that the inequality

$$m\left( \{ x \in \mathbb{R} : F(x) > \lambda \} \right) \leq \frac{2}{\lambda} \int_{\mathbb{R}} |f(x)| \, \mathrm{d}m(x)$$

holds for all $\lambda > 0$.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ a measurable function. For each $\lambda > 0$ define

$$m(\lambda) = \mu\left( \{ x \in X : |f(x)| > \lambda \} \right) \ .$$

Prove that the identity

$$\int_X |f(x)|^p \, \mathrm{d}\mu(x) = p \int_0^\infty \lambda^{p-1} m(\lambda) \, \mathrm{d}\lambda$$

holds for each positive real number $p$.

5. Suppose that $\{h_n\}$ is a sequence of positive continuous functions on the unit cube $Q_1 = \{x \in \mathbb{R}^d : 0 \le x_j \le 1,\ 1 \le j \le d\}$, and that $\mu$ is a positive Borel measure on $Q_1$. Let $m$ denote Lebesgue measure and assume
   (i) $\lim_{n\to\infty} h_n(x) = 0$ a.e. with respect to $m$
   (ii) $\int_{Q_1} h_n \, \mathrm{d}x = 1$ for all $n$
   (iii) $\lim_{n\to\infty} \int_{Q_1} f h_n \, \mathrm{d}x = \int_{Q_1} f \, \mathrm{d}\mu$ for every continuous function $f$ on $Q_1$.

   Prove or give a counterexample to the statement $\mu \perp m$.

**Part 2: Complex Analysis**

1. Suppose that $F(z)$ and $G(z)$ are entire functions having no common zeros. Prove that there exist entire functions $\varphi(z)$ and $\psi(z)$ such that
$$F(z)\varphi(z) + G(z)\psi(z) = 1$$
for all $z \in \mathbb{C}$.

2. Suppose that $f_1(z), f_2(z), \cdots, f_N(z)$ are continuous complex valued functions on the closed unit disk $\bar{\Delta} = \{z \in \mathbb{C} : |z| \le 1\}$ and analytic in the interior $\Delta$. Let $p$ be a real number greater than one. Prove that the function
$$g(z) = |f_1(z)|^p + \cdots + |f_N(z)|^p$$
takes its maximum value on the boundary of $\bar{\Delta}$. That is,
$$\sup\{g(z) : |z| \le 1\} = \sup\{g(z) : |z| = 1\} .$$

3. Suppose that $f(z)$ is analytic in the upper half plane $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ and satisfies $|f(z)| \le 1$ at each point $z$ in $\mathcal{H}$. Determine a positive constant $C$ such that $|f'(i)| \le C$ or show that such a constant $C$ does not exist.

4. Let $K \subseteq \mathbb{R}$ be a compact set with positive Lebesgue measure. Define $f : (\mathbb{C} \setminus K) \to \mathbb{C}$ by
$$f(z) = \int_K \frac{1}{t - z} \, \mathrm{d}m(t) .$$
   (i) Prove that $f$ is holomorphic on $\mathbb{C} \setminus K$.
   (ii) Prove that $f$ cannot be extended by analytic continuation to an entire function
   (iii) Show that $\lim_{|z|\to\infty} z f(z)$ exists and determine its value.

5. Use a rigorous argument to determine the value of the contour integral
$$\int_{\gamma_L} \frac{1}{z(z+1)} \, \mathrm{d}z$$
where $\gamma_L$ is the vertical line $\{z = -\frac{1}{2} + iy : -\infty < y < \infty\}$ with orientation in the positive $y$-direction.