

PRELIMINARY EXAMINATION IN ANALYSIS

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Part 1: Real Analysis.

1. Assume that $(X, \mathcal{A}, \lambda)$ is a measure space and $f \in L^1(X, \mathcal{A}, \lambda)$ satisfies $\|f\|_1 = \int_X |f(x)| d\lambda(x) = 1$. Prove that

$$\int_E \log |f(x)| d\lambda(x) \leq -\lambda(E) \log \lambda(E)$$

for all subsets $E \in \mathcal{A}$ such that $0 < \lambda(E) < \infty$.

2. Let $E \subset [0, 1]$ be measurable with $|E| > 0$.
- a) Show that $(\chi_E * \chi_E)(x)$ is a continuous function.
- b) If $E + t$ denotes the set $E + t = \{y, y - t \in E\}$ show that:
Given $E, \exists \varepsilon > 0$ such that

$$|E \cap E + t| > 0 \quad \forall |t| < \varepsilon$$

3. Consider the integral

$$h(x) = \int_0^\infty (x + y)^{-1} f(y) dy$$

defined for $x > 0$ and $f \in L^1(0, \infty)$. Show that h is infinitely differentiable away from the origin. Prove that $h' \in L^1[a, \infty)$ for any $a > 0$. What happens if we try to set $a = 0$?

4. Given the three measures in $[-2, 2]$,
- a) $\mu_1(E) = |E|$.
- b) $\mu_2(E) = |E \cap (0, 1)|$.
- c) $\mu_3(E) = \int_E x^2 dx$.

write the Lebesgue decomposition of each μ_i with respect to the others

$$d\mu_i = f d\mu_j + \sin \rho(\mu_i/\mu_j)$$

and compute $\int f d\mu_j$.

5. a) Give an example, in $[0, 1]$ of a sequence of functions f_k such that $\|f_k\|_{L^1} \rightarrow 0$, but $f_k \not\rightarrow 0$ a.e.
 b) If $\|f_k\|_{L^1} \leq 2^{-k}$, show that $f_k \rightarrow 0$ a.e.

Part 2: Complex Analysis.

1. Let $\Omega \subseteq \mathbb{C}$ be open, simply connected, and not empty. Show that if $f : \Omega \rightarrow \mathbb{C}$ is holomorphic and has no zeros in Ω , then there exists an holomorphic function $g : \Omega \rightarrow \mathbb{C}$ such that $f(z) = \exp\{g(z)\}$ at each point z in Ω .
 2. Let $p(z)$ be a polynomial with complex coefficients and degree $N \geq 1$. Let

$$\|p\|_{\infty} = \sup\{|p(z)| : z \in \mathbb{C}, |z| \leq 1\} .$$

Prove that

$$|p(w)| \leq \|p\|_{\infty} \max\{1, |w|\}^N \text{ for all } w \in \mathbb{C} .$$

3. Let $g(z) = u + iv$ be holomorphic in $(z) = x + iy$

$$D = \{|z| < 1, y > 0\}$$

and continuous in \overline{D} . Assume that

- a) $u = 0$ for $y = 0$ and $x > 0$
 b) $v = 0$ for $y = 0$ and $x < 0$

Show that:

$$\frac{|g(z)|}{|z|^{1/2}} \text{ is bounded in } \overline{D} .$$

4. Assume that $f : [0, \infty) \rightarrow \mathbb{C}$ belongs to $L^p([0, \infty), m)$, where $1 < p < \infty$ and m denotes Lebesgue measure. Define $F : \{z \in \mathbb{C} : 0 < \Re(z)\} \rightarrow \mathbb{C}$ by

$$F(z) = \int_0^{\infty} f(t) \exp\{-tz\} dm(t) .$$

Prove that F is holomorphic on $\{z \in \mathbb{C} : 0 < \Re(z)\}$.

5. Let f be holomorphic in D , with
 a) $|f| \leq |y|^{-1/2}$
 b) $\lim_{r \rightarrow 1} f(re^{i\theta}) = 0$ for any θ in $[0, 2\pi]$.
 Show that $f \equiv 0$