

APPLIED MATHEMATICS PRELIMINARY EXAMINATION
January 9, 2003

Work 4 of the following 5 problems.

1. Let V be a real Hilbert space and let $f \in V'$ be a continuous linear functional on V . Define the quadratic function $\varphi : V \rightarrow \mathbb{R}$ by

$$\varphi(v) = \frac{1}{2}\|v\|^2 - f(v)$$

for $v \in V$, where the norm is that of V .

(a) Show that φ is bounded below and that there is a unique point $u \in V$ for which φ realizes its minimum value; that is, for any $v \in V, v \neq u$,

$$\varphi(u) < \varphi(v).$$

(b) Show that at the minimum point u , we have

$$(u, v)_V = f(v) \quad \forall v \in V.$$

(c) Show that the map $f \mapsto u$ is *continuous* from V' to V .

(d) Use appropriate choices of V , $f(\cdot)$, and $\varphi(\cdot)$ to show that there is a unique weak solution to the boundary-value problem

$$-\Delta u + u = F \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded domain in \mathbb{R}^n . Specify carefully the space V , the assumptions on F , and explain what is meant here by a weak solution of (1).

(e) Suppose we have the following regularity result: if $F \in L^2(\Omega)$, then the solution of (1) satisfies $u \in H^2(\Omega)$. Show that the map $F \mapsto u$ is *continuous* from $L^2(\Omega)$ to $H^2(\Omega)$.

2. Consider the first-order differential equation

$$u'(t) + u(t) = \cos(u(t)) \tag{2a}$$

posed as an initial-value problem for $t > 0$ with initial condition

$$u(0) = u_0. \tag{2b}$$

(a) Use the contraction-mapping theorem to show that there is exactly one solution u of the initial-value problem (2) corresponding to any $u_0 \in \mathbb{R}$.

(b) Prove that there is a number ξ such that $\lim_{t \rightarrow \infty} u(t) = \xi$ for any solution u of (2a), independently of the value of u_0 .

3. Denote by $\mathcal{D}'(\Omega)$ the space of *distributions* on Ω .
- (a) Show that $T(\varphi) = \sum_{n=0}^{\infty} \varphi(n)$ defines $T \in \mathcal{D}'(\mathbb{R})$.
- (b) Show that the series $T = \sum_{n=0}^{\infty} a_n e^{nix}$ converges and defines $T \in \mathcal{D}'(\mathbb{R})$ if there is an $A \geq 0$ and an $N \geq 0$ such that $|a_n| \leq A|n|^N$.
- (c) Evaluate the derivatives $D(H(\cdot)\sin(\cdot))$ and $D^2(H(\cdot)\sin(\cdot))$ in $\mathcal{D}'(\mathbb{R})$ and verify your answer.
- (d) Show that $T(\varphi) = |\varphi(0)|$ is *not* in $\mathcal{D}'(\mathbb{R})$.
4. Let V and W be Hilbert spaces, and let $\mathcal{B} : V \rightarrow W'$ be continuous and linear.
- (a) Define the *adjoint* operator $\mathcal{B}' : W \rightarrow V'$ by $\mathcal{B}'w(v) = \mathcal{B}v(w)$, $\forall v \in V, w \in W$. Show that \mathcal{B}' is continuous and that its adjoint is given by $\mathcal{B}'' = \mathcal{B}$.
- (b) Let U be a subset of W . The *annihilator* of U is the set of functionals given by

$$U^\circ \equiv \{f \in W' : f(w) = 0 \forall w \in U\}.$$

Show that U° is a closed subspace of W' .

(c) Show that the closure of the range of \mathcal{B} is the annihilator of the kernel of \mathcal{B}' , that is,

$$\overline{\mathcal{B}(V)} = (\text{Ker } \mathcal{B}')^\circ.$$

(d) If there exists an $\alpha > 0$ such that

$$\sup_{w \in W} \frac{\mathcal{B}v(w)}{\|w\|_W} \geq \alpha \|v\|_V \quad \forall v \in V,$$

show that \mathcal{B} is an isomorphism of V onto $(\text{Ker } \mathcal{B}')^\circ$.

5. Let $c \in \mathbb{R}$ and $F(\cdot) \in L^2(0, 1)$ be given.
- (a) Discuss the solvability of the boundary-value problem

$$-(u'(x))' + cu(x) = F(x), \quad 0 < x < 1, \quad (3a)$$

$$u(0) = 0, \quad u'(1) = 0. \quad (3b)$$

Let $G(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ be a Lipschitz and monotone function, and denote by $g(\cdot)$ its antiderivative, *i.e.*, $g'(x) = G(x)$, $x \in \mathbb{R}$, and $g(0) = 0$.

(b) Show that the functional

$$J(u) = \frac{1}{2} \int_0^1 u'(x)^2 dx + \int_0^1 g(u(x)) dx - \int_0^1 F(x)u(x) dx$$

is Frechet differentiable on $H^1(0, 1)$.

(c) Show that there exists a unique solution of the boundary-value problem

$$-(u'(x))' + G(u(x)) = F(x), \quad 0 < x < 1, \quad (4a)$$

$$u(0) = 0, \quad u'(1) = 0. \quad (4b)$$