PRELIM EXAM IN APPLIED MATHEMATICS

Spring 2003

Solve 5 out of the 6 problems

1. Consider the function $A_p : L^p([0,1]) \to L^1([0,1])$ that to a function $f \in L^p(0,1)$ assigns the function in $L^1(0,1)$

$$(A_p f)(x) = |f(x)|^p$$

1.a) For which values of p is this operator continuous?

1.b) For which values of p is this operator strongly differentiable?

1.c) In the cases that the operator is strongly differentiable, compute the strong derivative and verify that it satisfies the definition of the strong derivative.

1.d) For which values of p is the operator twice differentiable?

1.e) In the cases that the operator is twice strongly differentiable, compute the second strong derivative.

2. Let A be a bounded linear operator from a Banach space X to itself (i.e. $A \in B(X,X)$).

Suppose $||A|| < 1$.

2.a) Show that $||A^2|| \leq ||A||^2$. In case that A is boundedly invertible show that $||A^{-1}|| \geq ||A||^{-1}$.

2.b) Show that $(\mathrm{Id} - A)^{-1}$ is a bounded operator and

$$(\mathrm{Id} - A)^{-1} = \mathrm{Id} + A + A^2 + \cdots$$

2.c) Show that

$$\frac{1}{1 + ||A||} \leq ||(\mathrm{Id} - A)^{-1}|| \leq \frac{1}{1 - ||A||}$$

3.

3.a) For an integer n, consider the real functions

$$f_n(x) = n^2 \chi_{-1/n^2,0} - n^2 \chi_{0,1/n^2,0}$$

Compute $\lim_{n\to\infty}$ in the sense of distributions.

3.b) For which values of α does

$$\varphi \to \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} |x|^{-\alpha} \varphi(x)\, dx$$

define a distribution?

3.c) For which values of α does

$$\varphi \to \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \operatorname{sign}(x)|x|^{-\alpha}\varphi(x)\, dx$$

define a distribution?

4. Let Ψ be a $C^\infty(\mathbf{R}^n)$ non-negative function with compact support. such that $\int \Psi = 1$. Define $\Psi_\varepsilon(x) = \varepsilon^{-n}\Psi(\varepsilon x)$.

Show that

$$f_\varepsilon = f * \Psi_\varepsilon$$

converges to f in the following senses:

4.a) $f_n(x) \to f(x)$ At every x which is a point of continuity of f.

4.b) In L^p sense for every $p \in [1, \infty)$.

4.c) Is the result in 4.b) also true in L^∞?

5. Consider the variational problems for pairs of differentiable functions in the interval $[0, 2\pi]$ $x(0) = y(0) = 0$, $x(2\pi) = y(2\pi) = 0$.

$$J_\pm(x, y) = \int_0^1 dt\ (x'(t)^2 \pm y'(t)^2 + 2x(t)y(t))$$

5.a) Compute the Euler-Lagrange equations.

5.b) Does the functional reach a minimum?

6. Consider the functions f_n defined by $f_0 = 1/2\chi_{[0,1/3]} + 1/2\chi_{[2/3,1]}$ and

$$f_{n+1}(x) = 3/2 f_n(3x) + 3/2 f_n(3(x - 2/3)).$$

6.a) Show that $\int f_n = 1$.

6.b) Show that f_n converges in the weak sense of measures.

Hint: One way is to consider the measure of intervals of the form $[k3^{-l}, (k+1)3^{-l}]$, k, l natural numbers.

6.c) Show that f_n converges in the sense of distributions.

6.d) Compute the Fourier transform of the the limit of f_n.