APPLIED MATHEMATICS PRELIMINARY EXAMINATION

August 22, 2000

Work 5 of the following 6 problems.

1. Let H be a Hilbert space with closed subspaces M and N such that $M \cap N = \{0\}$ and $H = M + N$. Let $P_N : H \to N$ denote the orthogonal projection operator.
   (a) Define what we mean by the space $N^\perp$.
   (b) Define the operator $Q : M \to N^\perp$ as $I - P_N$. Show that $Q \equiv P_{N^\perp} \equiv P_N^\perp$ when the latter two operators are restricted to M. Prove that Q is both injective and surjective.
   (c) Prove that there is an $\alpha > 0$ such that

$$\|x - P_N x\| \geq \alpha \|x\| \qquad \forall x \in M.$$

2. Let f be a real-valued function of a real variable.
   (a) Define what it means for f to be asymptotic to the series $\sum_{n=0}^{\infty} \frac{a_n}{x^n}$ as $x \to +\infty$.
   (b) Show that $f(x) = \mathrm{e}^{-x}$ is asymptotic to the null expansion where $a_n = 0$ for all n.
   (c) Show by example that even if f is infinitely differentiable, formal differentiation of its asymptotic expansion need not lead to a valid asymptotic expansion. That is, it is not necessarily true that

$$f(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{x^n} \qquad \text{as } x \to \infty \tag{1}$$

   implies

$$f'(x) \sim -\sum_{n=1}^{\infty} n \frac{a_n}{x^{n+1}} \qquad \text{as } x \to \infty. \tag{2}$$

   (d) Give hypotheses under which (1) implies (2) is valid and prove your result.

3. Let X be a Banach space.
   (a) Suppose Y is a proper closed subspace of the subspace Z of X ($Y \subset Z$ and $Y \neq Z$). Show that there is a $z \in Z$ such that

$$\|z\| = 1, \qquad \text{and} \qquad \operatorname{dist}\{z, Y\} \geq \frac{1}{2}.$$

   (b) Let $A : X \mapsto X$ be a compact linear operator. Show that 0 is the only possible limit point of the spectrum $\sigma(A)$ of A. (Hint: Consider the set $\sigma(A) \cap \{z : |z| > \epsilon\}$ where $\epsilon > 0$.)

4. Let V be a real Hilbert space and let $f \in V'$ be a continuous linear functional on V. Define the quadratic function $\varphi : V \to \mathbb{R}$ by

$$\varphi(v) = \frac{1}{2}\|v\|^2 - f(v)$$

for $v \in V$, where the norm is that of V.

(a) Show that φ is bounded below and that there is a unique point $u \in V$ for which φ realizes its minimum value; that is, for any $v \in V, v \neq u$,

$$\varphi(u) < \varphi(v).$$

(b) Show that at the minimum point u, we have

$$(u, v)_V = f(v) \quad \forall v \in V.$$

(c) Use appropriate choices of φ and V to show that there are weak solutions to the boundary-value problem

$$-\Delta u + u = F \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$. Specify carefully the function classes used and explain what is meant here by a weak solution.

5. Consider the first-order differential equation

$$u'(t) + u(t) = \cos(u(t)) \tag{3}$$

posed as an initial-value problem for $t > 0$ with initial condition

$$u(0) = u_0. \tag{4}$$

(a) Use the contraction-mapping theorem to show that there is exactly one solution u of (3–4) corresponding to any $u_0 \in \mathbb{R}$.

(b) Prove that there is a number ξ such that $\lim_{t\to\infty} u(t) = \xi$ for any solution u of (3), independently of the value of u_0.

6. Let $f(x) = |x|^{-1}$ on $\mathbb{R}$. Consider T_f defined by

$$T_f(\phi) = \int_{-\infty}^{\infty} |x|^{-1}\phi(x)\, dx.$$

(a) Show that T_f is not well defined for $\phi \in \mathcal{D}$.

(b) Show that T_f is defined for all $\phi \in \mathcal{D}$ which vanish at the origin.

(c) Show that there is a $T \in \mathcal{D}'$ such that T agrees with T_f on test functions that vanish at the origin.