APPLIED MATHEMATICS PRELIMINARY EXAMINATION
August 24, 2001

Work any 5 of the following 6 problems.

1. Let X be a normed linear space.
(**a**) Define weak convergence in X. Discuss and justify uniqueness properties of weak convergence in X.
(**b**) Show that strong convergence in X implies weak congergence in X.
(**c**) Show, with a counterexample, that weak convergence does not imply strong convergence.
(**d**) Show that if X is finite dimensional then strong convergence is equivalent to weak convergence.

2. Let $\Omega = [0,1]^n \subset \mathbb{R}^n$ and

$$C_P^\infty(\Omega) = \{u \in C^\infty(\Omega) : u \text{ is periodic on } \Omega\} .$$

Let $H_P^1(\Omega)$ be the completion of $C_P^\infty(\Omega)$ in $H^1(\Omega)$, and let

$$H_{P,0}^1(\Omega) = \left\{ u \in H_P^1(\Omega) : \int_\Omega u(x)\, dx = 0 \right\} .$$

(**a**) Show that $H_{P,0}^1(\Omega)$ is a Hilbert space with the $H^1(\Omega)$ inner product.
(**b**) Prove that there is some constant $C > 0$ such that for any $u \in H_{P,0}^1(\Omega)$,

$$\int_\Omega |u(x)|^2\, dx \le C \int_\Omega |\nabla u(x)|^2\, dx .$$

(**c**) For some fixed $u_0 \in H_P^1(\Omega)$ and $f \in L^2(\Omega)$, consider the variational problem: Find $u \in H_{P,0}^1(\Omega) + u_0$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx , \quad \forall v \in H_{P,0}^1(\Omega) .$$

Prove that there is a unique solution.
(**d**) Show that (c) is equivalent to the boundary value problem: Find $u \in H_{P,0}^1(\Omega) + u_0$ such that

$$-\Delta u = f .$$

3. Show that for $y \in \mathbb{R}^2$ fixed, $\frac{1}{2\pi} \ln|x - y|$ is locally integrable in $\mathbb{R}^2$, i.e. it is a function in $L_{1,loc}(\mathbb{R}^2)$; and that it is a fundamental solution of $\Delta u = \delta_y$, where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ is the Laplace operator.

4. Find the leading order large t behavior of the integral:

$$\int_{R^1} e^{-t^3x^2 + t^2 x}\, dx.$$

(Hint: complete the square in x or use series expansion)

5. Consider the functional

$$J[x,y] = \int_0^{\pi/2} \left[(x'(t))^2 + (y'(t))^2 + 2x(t)y(t)\right] dt$$

and the boundary conditions

$$x(0) = y(0) = 0 \quad \text{and} \quad x(\pi/2) = y(\pi/2) = 1 \ .$$

(**a**) Find the Euler-Lagrange equations for the functional.
(**b**) Find all extremals.
(**c**) Find a global minimum, if it exists, or show it does not exist.
(**d**) Find a global maximum, if it exists, or show it does not exist.

6. Set up and apply the contraction mapping principle to show that the boundary value problem ($\varepsilon > 0$):

$$\begin{cases} -u_{xx} + u - \varepsilon u^2 = f(x), \ x \in (0,+\infty), \\ u(0) = 1, \ u(+\infty) = 0, \end{cases}$$

where $f(x)$ is a smooth compactly supported function on $(0,+\infty)$, has a unique smooth solution if ε is small enough.