# APPLIED MATHEMATICS
# PRELIMINARY EXAMINATION

August 25, 2005, 1:00-4:00 p.m.

*Work any 5 of the following 6 problems.*

**1.** Let $\big(X, \|\cdot\|\big)$ be a Banach space, and let $\big(X', \|\cdot\|_{X'}\big)$ denote the corresponding dual space.

(**a**) State the Hahn-Banach Theorem for a normed linear space.

(**b**) Show that for every non-zero $x \in X$ there exists an $f_x \in X'$ such that

$$\|f_x\|_{X'} = 1 \quad \text{and} \quad f_x(x) = \|x\|.$$

(**c**) Show that if $f(x) = f(y)$ for all $f \in X'$, then $x = y$. You should first show that

$$\|x\| = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|_{X'}}, \quad \forall x \in X.$$

**2.** Let $1 \leq p < \infty$ and define, for each $r \in \mathbb{R}^d$, $T_r : L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d)$ by

$$T_r(f)(x) = f(x + r).$$

(**a**) Verify that indeed $T_r$ maps into $L_p(\mathbb{R}^d)$ and that $T_r$ is bounded and linear. What is the norm of $T_r$ ?

(**b**) Show that as $r \to s$, $\|T_r f - T_s f\|_{L_p} \to 0$. [Hint: Use that the set of continuous functions with compact support is dense in $L_p$ for $p < \infty$.]

**3.** Let $T$ be a bounded linear operator on a Banach space $X$ such that the sequence $I, T, T^2, ...$ is bounded. Show that if $\lambda \in \sigma(T)$, then $|\lambda| \leq 1$. [Hint: Investigate the limit when $N$ goes to infinity of $R_N x = \sum_{n=0}^{N} \frac{1}{\lambda^n} T^n x$.]

**4.** Suppose that $\Omega \subset \mathbb{R}^d$ is bounded and that $f_j$ and $g_j$ are bounded sequences in $H^1(\Omega)$. Moreover, suppose $f_j$ and $g_j$ converge weakly to $f$ and $g$, respectively, in $H^1(\Omega)$. Let $D = \partial/\partial x_1$.

(**a**) Show that $D(f_j g_j)$ converges to $D(fg)$ as a distribution.

(**b**) Find all $p$, $1 \leq p < \infty$, such that $D(f_j g_j) = f_j D g_j + g_j D f_j$ is bounded in $L_p(\Omega)$.

(**c**) Show that for any weakly convergent subsequence $D(f_{j_k} g_{j_k})$ in $L_p(\Omega)$, the limit must be $D(fg)$.

**5.** Show that if $g \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ is a complex valued function, then the complex valued solution $u = u(t, x)$ of the initial valued problem (Schroedinger equation)

$$iu_t + \Delta u = 0, \quad x \in \mathbb{R}^d,\ t \in \mathbb{R}^+,$$
$$u(x, 0) = g(x), \quad x \in \mathbb{R}^d,$$

preserves the $L_2$-norm in space of the initial state; that is,

$$\|u(t,\cdot)\|_{L_2(\mathbb{R}^d)} = \|g\|_{L_2(\mathbb{R}^d)} \quad \text{for all } t > 0.$$

**6.** Suppose that $\Omega \subset \mathbb{R}^d$ is bounded and $g \in L_1(\Omega)$. Show that the solution $u(t,\cdot) \in L_1(\Omega) \cap H^1(\Omega)$ of the initial value problem (Heat equation)

$$\begin{aligned} &u_t - \Delta u = 0, && x \in \Omega,\ t \in \mathbb{R}^+, \\ &\nabla u \cdot \nu = 0, && x \in \partial\Omega,\ t \in \mathbb{R}^+, \\ &u(x,0) = g(x), && x \in \partial\Omega, \end{aligned}$$

satisfies the following.

(**a**) For all times $t$, $\displaystyle\int_\Omega u(x,t)\,dx = \int_\Omega g(x)\,dx$.

(**b**) If $\displaystyle\int_\Omega g(x)\,dx = 0$, the $L_2$-norm of $u$ in space decays in time. What is the decay rate for $\|u(t,\cdot)\|_{L_2(\Omega)}$?