APPLIED MATHEMATICS
PRELIMINARY EXAMINATION

January 12, 2006, 1:00-4:00 p.m.

Work any 5 of the following 6 problems.

1. Let V and W be Banach spaces, and $A : V \to V^*$ and $B : W \to V^*$ be continuous linear operators. Suppose that A is surjective and that it is *coercive,* meaning that there is some $\alpha > 0$ such that

$$\langle Av, v\rangle \geq \alpha \|v\|^2, \qquad \forall v \in V.$$

Also suppose that there is some $\beta > 0$ such that

$$\inf_{0 \neq w \in W} \sup_{0 \neq v \in V} \frac{\langle Bw, v\rangle}{\|w\| \, \|v\|} \geq \beta > 0.$$

For $f \in V^*$ and $g \in W^*$, consider the problem

$$\begin{cases} Au + Bp &= f, \\ B^*u &= g. \end{cases}$$

(**a**) Prove that A is injective, and thus has an inverse.

(**b**) Prove that A^{-1} is continuous and coercive.

(**c**) If a solution (u, p) to the problem exists, prove that it is unique.

(**d**) If $B^*A^{-1}B : W \to W^*$ is surjective, prove that there is a unique solution to the problem.

2. Consider the functions F_n and f_n defined on $\mathbb{R}$ by

$$F_n(x) = \begin{cases} 1, & \text{if} \quad |x| \leq n, \\ 0, & \text{if} \quad |x| > n, \end{cases} \qquad \text{and} \qquad f_n(x) = \frac{\sin nx}{x}.$$

(**a**) Show that $\widehat{F}_n = \sqrt{\frac{2}{\pi}} f_n$ where $\widehat{F}_n(\xi) = \frac{1}{\sqrt{2\pi}} \int F_n(x) e^{-i\xi x}\, dx$ is the Fourier transform of F_n.

(**b**) Prove that $F_n \to 1$ in $\mathcal{D}'$.

(**c**) Show that f_n is a distribution.

(**d**) Prove that $f_n \to \pi\delta_0$ in $\mathcal{D}'$.

3. Consider $f \in L_p([0,1])$ for $1 < p < \infty$. Define q by $1/q + 1/p = 1$ and define $T_f : L_q([0,1]) \to L_q([0,1])$ by

$$T_f g(x) = \int_0^x f(y) g(y)\, dy, \qquad \forall x \in [0,1].$$

(**a**) Show that T_f is a bounded operator from $L_q([0,1])$ to $L_q([0,1])$.

(**b**) State what it means for T_f to be compact.

(**c**) Show that T_f is a compact operator. (Hints: The unit ball in L_q is weakly compact. You may need to use the Dominated Convergence Theorem to pass from pointwise convergence to convergence in L_q).

4. Consider the one-dimensional boundary-value problem

$$\text{(BVP)} \quad \begin{cases} -u'' + u &= f_\varepsilon, \quad a < x < b, \\ u(a) - \beta_a u'(a) &= 0, \\ u(b) + \beta_b u'(b) &= 0, \end{cases}$$

where $\beta_{a,b} \neq 0$ are constants, $\varepsilon > 0$ is a parameter, and $f_\varepsilon \in C([a, b])$ for each ε.

(**a**) Write a variational or weak form of (BVP) using test functions in $H^1([a, b])$.

(**b**) Show that, if $\beta_{a,b} > 0$, then the weak form has a unique solution $u_\varepsilon \in H^1([a, b])$ for each $\varepsilon > 0$. Does your proof extend to the case when the sign of β_a and/or β_b are changed? Why or why not?

(**c**) Suppose that $f_\varepsilon \to f_0$ in $L_2([a, b])$ as $\varepsilon \to 0$ where f_0 is a discontinuous function. Show that the solutions u_ε from part (b) converge in an appropriate sense to a limiting function u_0 as $\varepsilon \to 0$. In what sense is u_0 a solution of (BVP)? Could it ever be a classical solution in this case?

5. Consider a functional $F : V \to \mathbb{R}$ where

$$V = \{y(x) \in C^1([0, 1]) \mid y(0) = 1\}, \qquad F(y) = \int_0^1 L(x, y, y')\, dx + G(y(1)).$$

Here L and G are smooth functions.

(**a**) Introduce an appropriate space of admissible variations and derive the Euler-Lagrange equations that any sufficiently smooth extremal y must satisfy. What boundary condition must y satisfy at $x = 1$?

(**b**) Find all smooth extremals of F in the case when $L(x, q, p) = p^2 - 2xqp$ and $G(q) = q^2$.

6. Consider the initial-value problem

$$\text{(IVP)} \quad \begin{cases} u' &= f(u), \quad 0 < t < T, \\ u(0) &= u_0, \end{cases}$$

where $u(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given continuous function.

(**a**) Introduce an appropriate mapping $F : C(0, T) \to C(0, T)$ and show that u is a continuously differentiable solution of (IVP) if and only if it is a fixed-point of F.

(**b**) Suppose f is globally Lipschitz in the sense that there is a constant $L > 0$ such that

$$||f(x) - f(y)|| \leq L||x - y||, \qquad \forall x, y \in \mathbb{R}^n.$$

Show that (IVP) has a unique continuously differentiable solution u provided $T > 0$ is sufficiently small.

(**c**) Show that the unique solution from (b) can be extended to a unique solution on the interval $0 < t < \infty$.