PRELIMINARY EXAMINATION IN APPLIED MATHEMATICS

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Instructions: Solve 4 of the following 6 problems. Notice that each problem has several parts.

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1 (a). Define the term "compact linear transformation".

1 (b). Prove that if L is a compact linear transformation of a normed linear space X into itself, and if λ is a non-zero complex number, then the subspace $\{x \in X : Lx = \lambda x\}$ is finite-dimensional.

2 (a). State and prove Banach's contraction mapping theorem for an operator $T : M \to M$ on a complete metric space M.

2 (b). Use Part (a) to show that if $T : M \to M$ is an operator for which the composition T^N is a contraction for some $N \geq 1$, then T has a unique fixed point.

2 (c). Suppose that a Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$ and a number $u_0 \in \mathbb{R}$ are given. Use either Part (a) or Part (b) above to prove that there exists a unique solution $u(\cdot)$ of the initial-value problem

$$u'(t) = f(u(t)) \ , \qquad 0 < t < 1 \ , \qquad u(0) = u_0 \ .$$

3 (a). Let $\{u_1, u_2, \dots\}$ be an orthonormal base for $L^2([a, b])$. Prove that, for all t in the interval $[a, b]$,

$$\sum_{n=1}^{\infty} \left| \int_a^t u_n(s)\, ds \right|^2 = t - a \ .$$

3 (b). (The Theorem of the Fredholm Alternative) Let A be a bounded linear operator on a Hilbert space. Assume that the range of A is closed. Prove that the equation $Ax = b$ has a solution if and only if b is orthogonal to the kernel of A^*. (Suggestion: Prove first that the kernel of A^* is the orthogonal complement of the range of A.)

4. Let X be a Hilbert space, and let A be a linear map of X into X. (Do not assume that A is continuous.) Let B be a map of X into X such that $\langle Ax, y \rangle = \langle x, By \rangle$ for all x and y in X. (Do not assume that B is linear nor that it is continuous.) Prove that (a) B is linear; (b) B is continuous; and (c) A is continuous.

5 (a). Show how the approximate formulas

$$f'(t) \approx \frac{f(t+h) - f(t-h)}{2h}$$
$$f''(t) \approx \frac{f(t+h) - 2f(t) + f(t-h)}{h^2}$$

can be used to obtain an approximate solution of the following two-point boundary-value problem:

$$u''(t) + a(t)u'(t) + b(t)u(t) = c(t) \qquad (0 < t < 1)$$
$$u(0) = 0 \qquad u(1) = 0$$

Here, a, b, and c are continuous functions on $[0, 1]$.

5 (b). Prove that if a uniform mesh is used in the numerical procedure of Part (a), that is

$$t_i = ih \qquad i = 0, 1, \ldots, n+1 \qquad h = 1/(n+1)$$

then the resulting system of linear equations will be nonsingular, provided that $b(t) < 0$ and h is sufficiently small.

6 (a). Let

$$H = \{u \in L^2[0,1] : u' \in L^2[0,1]\}$$

in which u' signifies the distribution derivative. In H we use the norm

$$\|u\| = \left[\int_0^1 \{|u(t)|^2 + |u'(t)|^2\}\, dt\right]^{1/2}$$

Define $F : H \to \mathbb{R}$ by the equation

$$F(u) = \int_0^1 \{[u'(t)]^2 + a(t)[u(t)]^2 - 2b(t)u(t)\}\, dt$$

in which $a \in L^\infty[0,1]$, $b \in L^2[0,1]$, and $a(t) \geq \varepsilon > 0$ almost everywhere on $[0,1]$. Compute the Frechet derivative of F.

6 (b). Show that the equation $F'(u) = 0$ is equivalent to a Sturm-Liouville problem.

6 (c). Prove that there is a unique solution of the equation $F'(u) = 0$.