PRELIMINARY EXAMINATION IN APPLIED MATHEMATICS

August 1998

Instructions: Solve any five problems.

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1. a) Define what it means for a set X to be a normed linear space.
   b) When is a normed linear space a Banach space? When is a Banach space a Hilbert space?
   c) Give an example of a Banach space that is not a Hilbert space, and prove this asssertion about your example.
   d) State the Open Mapping Theorem. Assuming the Open Mapping Theorem is valid, prove the Closed-Graph Theorem.

2. Let X, Y be Banach spaces and let $A : X \to Y$ be a linear operator.
   a) Define what it means for A to be bounded and prove this is equivalent to continuity. Show that the set of all such operators forms a Banach space $B(X, Y)$ in its own right.
   b) Suppose A is in $B(X, Y)$ and $\|A\|_{B(X,Y)} < 1$. Show that $I - A$ is invertible, where $I : X \to X$ is the identity map, and that
$$\frac{1}{1 + \|A\|} \leq \|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|} \ .$$

3. Let X, Y be Banach and let $U \subset X$ be an open subset. Let $F : U \to Y$ be a not necessarily linear operator.
   a) Define what it means for F to be Fréchet differentiable at $x \in U$.
   b) Let $X = C([0, 1])$ be the space of bounded, continuous, real-valued functions on $[0, 1]$. For $u \in X$, let
$$F(u)(x) = \int_0^1 K(x, y) f(u(y))\, dy$$
   where $K : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous and f is a C^1-mapping of $\mathbb{R}$ into $\mathbb{R}$. Find the Fréchet derivative of F at a point $u_0 \in X$ and prove it satisfies the definition.
   c) For the operator F above, is the mapping $u \mapsto DF(u)$ continuous? Further prove this, or offer a counterexample.

4. Let $\mathcal{D}'$ denote the space of distributions and $\mathcal{D}$ the space of test functions.

   a) If $T \in \mathcal{D}'$ and $\phi \in \mathcal{D}$, define the convolution $T * \phi$. Show that, for each multiindex α

$$D^\alpha(T * \phi) = (\partial^\alpha T) * \phi = T * D^\alpha \phi$$

   b) If $T \in \mathcal{D}'$ and $\phi \in \mathcal{D}$, show that $T * \phi \in C^\infty(\mathbb{R}^n)$.

   c) For 2π-periodic function $(f * g)(x) = \int_0^{2\pi} f(y)g(x-y)\,dy$ compute the convolution of $f(x) = \sin x$ and $g(x) = \cos x$.

5. Implicit Function Theorem.

   a) State the Implicit Function Theorem.

   b) Let X and Y be Banach spaces, let F and G take X to Y be C^1 on X, and let $H(x, \varepsilon) = F(x) + \varepsilon G(x)$ for $\varepsilon \in \mathbb{R}$. If $H(x_0, \varepsilon_0) = 0$ and $DF(x_0)$ is invertible, show that there exists $x \in X$ such that $H(x, \varepsilon) = 0$ for ε sufficiently close to ε_0.

   c) For small ε, prove that there is a solution $w \in H^2(0, \pi)$ to $w'' = w + \varepsilon w^2$, $w(0) = w(\pi) = 0$.

6. Consider the functional $I(y) = \displaystyle\int_0^a [(y'(x))^2 - (y(x))^2]\,dx$, where $a > 0$ and $y \in C^1([0,1])$.

   a) Find the extrema of $I(y)$ when $a = 2\pi$ and $y(0) = y(2\pi) = 1$.

   b) Find the extrema of $I(y)$ when a is *not* a multiple of π, $y(0) = 0$ and $y(a) = 1$.

   c) Find the extrema of $I(y)$ subject to the constraint $\displaystyle\int_0^\pi (y(x))^2\,dx = 1$ when $a = \pi$ and $y(0) = y(\pi) = 0$. What is the best constant in the Poincaré inequality $\displaystyle\int_0^\pi f^2\,dx \le C \int_0^\pi f'^2\,dx$, for $f \in H_0^1(0, \pi)$?