Preliminary Examination in Topology: August 2003

Instructions: Do two questions from section A and two from section B.

Time Limit: 4 hours

Section A

1. Show that each closed connected surface is a covering of exactly one of the following surfaces:

$$\mathbf{RP}^2, \quad \mathbf{RP}^2\#\mathbf{RP}^2, \quad \mathbf{RP}^2\#\mathbf{RP}^2\#\mathbf{RP}^2.$$

2.(a) Compute, with complete justification, all the homology groups $H_*(S^4; \mathbf{Z})$.

(b) Let $a : S^4 \to S^4$ denote the antipodal map. Compute the Lefschetz number and the degree of a.

(c) Show that the only non-trivial group of homeomorphsims to act **freely** on S^4 is the cyclic group of order 2. (**Hint:** Use Part (b))

3.(a) A 2-disc D^2 is attached to a 2-torus T^2 by a map $f : S^1 = \partial D^2 \to T^2$. If $X = T^2 \cup_f D^2$ what are the possible homology groups $H_*(X; \mathbf{Z})$?

(b) A 3-disc D^3 is attached to a 2-torus T^2 by a map $g : S^2 = \partial D^3 \to T^2$. If $Y = T^2 \cup_g D^3$ what are the possible homology groups $H_*(Y; \mathbf{Z})$?

Section B

4. The space M of 2×2 real matrices can be identified with $\mathbf{R}^4$ by the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

(a) Prove that the subset $\mathrm{SL}(2, \mathbf{R}) = \{\mathrm{A} \in \mathrm{M} : \det\ \mathrm{A} = 1\}$ is a manifold.

(b) Find the critical values of of the **trace function** $\mathrm{tr} : \mathrm{SL}(2, \mathbf{R}) \to \mathbf{R}$, defined by $\mathrm{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{a} + \mathrm{d}$.

5. Let $f, i : \mathbf{RP}^1 \to \mathbf{RP}^2$ be given by

$$f([x_0 : x_1]) = [x_0^2 : x_1^2 : x_0 x_1] \quad \text{and} \quad i([x_0 : x_1]) = [x_0 : x_1 : 0].$$

(a) Compute the intersection number $I_2(f, i(\mathbf{RP}^1))$.

(b) Is f homotopic to i?

6. An **oriented line field** L on a manifold X assigns to each $x \in X$ an oriented 1-dimensional subspace L_x of $T_x X$, in such a way that X is covered by local parametrizations $\phi : U \to X$ under which L corresponds to a line field parallel to $\mathbf{R} \times \{0\} \subset \mathbf{R} \times \mathbf{R}^{n-1}$ (that is, for all $u \in U \subset \mathbf{R}^n$, $\mathrm{d}\phi_{\mathrm{u}}(\mathrm{e}_1) \in \mathrm{L}_{\phi(\mathrm{u})}$.)

(a) If a compact oriented manifold X admits an oriented line field, construct a nowhere-zero vector field on X.

(b) Of the following manifolds, which ones admit oriented line fields?

$$S^4, \quad S^3 \times S^1, \quad S^2 \times S^2$$