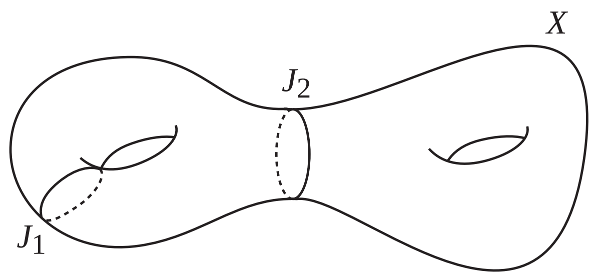
Topology Preliminary Examination
January 2003

Do two out of three in each part.

Part I.

1. Below is pictured a surface X that is a double torus with two curves drawn on it.
   (a) Does X retract to J_1? Prove your answer.
   (b) Does X retract to J_2? Prove your answer.



2. Let T_1 and T_2 be tori and J_1 and J_2 be homotopically trivial simple closed curves on T_1 and T_2 respectively. Let X be the quotient space obtained by identifying J_1 and J_2 by a homeomorphism.
   (a) Use Van Kampen's Theorem to compute the fundamental group of X.
   (b) Use the Mayer-Vietoris Theorem to compute all homology groups of X.
3. Let X equal the connected sum of three projective planes.
   (a) What spaces are 2-fold covers of X? In each case, describe a covering map.
   (b) Either prove the following or give a counterexample: Suppose $\pi : Y \to X$ is a 2-fold covering map of X, where Y is orientable. Then every embedding of a cylinder into X lifts to Y.

Part II.

1. Let $T^n = R^n/Z^n = S^1 \times S^1 \times \cdots \times S^1$ denote the n-torus.
   (a) (i) Give an explicit vector field on T^n whose zeroes (if any) are isolated.
   (ii) Compute the Euler Characteristic $\chi(T^n)$.
   (b) Let X be a smooth oriented manifold with an orientation-preserving local diffeomorphism $\varphi : T^n \to X$. List all possible values of $\chi(X)$, and prove your answer.
2. Let F_2 be a closed, oriented surface of genus 2.
   (a) Define two embeddings $T^2 = S^1 \times S^1 \to F_2 \times F_2$ whose images have nonzero intersection number. (Prove your answer.)
   (b) For some $x \in F_2$, let $F_2^* = F_2 - \{x\}$. Give three embeddings $T^2 \to F_2^* \times F_2^*$ such that no two of them are homotopic in any 4-manifold X containing $F_2^* \times F_2^*$ as an open subset. Prove your answer.
3. Let $\omega = zdx \wedge dy + (z^2 e^{xz} \ln(x^2+1) - y)dx \wedge dz + (x + y^2 z^3 \cos(y^2+z^2))dy \wedge dz$.
   (a) Is ω closed? Is ω exact?
   (b) Let B be the bullet-shaped surface-with-boundary formed by taking the union of the cylinder $x^2+y^2 = 1$, $-1 \le z \le 1$, with a hemisphere $x^2+y^2+(z-1)^2 = 1$, $z \ge 1$. Orient B such that $dx \wedge dy$ is positive at (0,0,2). Compute $\int_B \omega$.