**Preliminary Examination in Topology: August 2004**

**Instructions**: Do two questions from section A and two from section B. Explain your reasoning throughout and if you quote theorems, quote them completely.

**Time Limit**: 4 hours

**Section A**

**1.** Let's define the 3-hole connected sum of two closed connected surfaces, $M_1^2$, $M_2^2$, as follows: Let $D_1$, $D_2$ , $D_3$ be three disjoint disks in $M_1^2$ and let $E_1$, $E_2$, $E_3$ be three disjoint disks in $M_2^2$ . Then the "3-hole connected sum" of $M_1^2$ and $M_2^2$ is the space created by removing the interiors of $D_1$, $D_2$, and $D_3$ from $M_1^2$, removing the interiors of $E_1$, $E_2$, and $E_3$ from $M_2^2$, and identifying $\partial D_1$ with $\partial E_1$, $\partial D_2$ with $\partial E_2$, and $\partial D_3$ with $\partial E_3$ via homeomorphisms ("$\partial$" denotes boundary).

(a) Is the 3-hole connected sum well-defined up to homeomorphism? Explain.

(b) Describe all the surfaces that can result from a 3-hole connected sum of a Klein bottle $K_2$ with a genus 2 "double torus" $M_2$ . Explain your answers.

**2.** Consider the space $Y$, shown on the board, that consists of the union of a (hollow) genus$-2$ orientable surface and a disk.

(a) Compute, with complete justification, $\pi_1(Y)$.

(b) Compute, with complete justification, the homology groups $H_n(Y;\mathbb{Z})$ for $n = 0$, 1, 2, 3.

**3.**

(a) Describe a space $X$ such that $\pi_1(X)$ is isomorphic to $(\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}$.

(b) Show that $(\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}$ has a subgroup of index 2 isomorphic to $(\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z} * \mathbb{Z}$.

(c) Describe a non-normal subgroup of $(\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}$ of index 3.

**Section B**

**4.** Consider the function $f : S^2 \to S^2$ defined by $f(x, y, z) = (y, z, x)$.

(a) Find the fixed points of $f$.

(b) Compute the local Lefschetz numbers.

(c) Is $f$ homotopic to the identity map?

(d) Let $Y$ be the graph of $f$ and let $\Delta$ be the diagonal in $S^2 \times S^2$ . Compute the intersection number $I(Y, \Delta)$ .

**5.** Let $X \subset \mathbb{R}^n$ be a closed (but NOT compact) subset that is a smooth manifold (without boundary). Let $K \subset X$ be an arbitrary compact subset. Show that there is a compact, smooth manifold-with-boundary $Y$ such that $K \subset Y \subset X$.

**6.** Let $S^1$ be the unit circle in $\mathbb{R}^2$, and consider a map $f : S^1 \to S^1$ of degree $n$. Let $d\theta = xdy - ydx$, where $x$ and $y$ are the usual coordinates in $\mathbb{R}^2$.

(a) Prove that $\int_{S^1} f^* d\theta = 2\pi n$.

(b) Identifying $\mathbb{R}^2$ with the complex numbers, let $g : S^1 \to S^1$ be defined by $g(z) = zf(z)$. Compute $g^*(d\theta)$ in terms of $f^*(d\theta)$.

(c) Show that $\deg(g) = \deg(f) + 1$.