PRELIMINARY EXAMINATION IN TOPOLOGY

August 2005   4 Hours

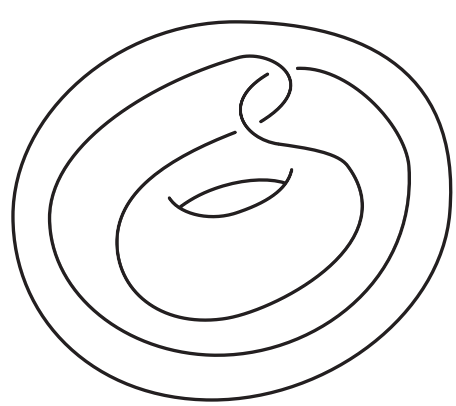
Instructions: Work 2 out of 3 problems in each part of the exam. Explain all of your work carefully.

Part 1

1. (a) Describe a CW decomposition for the Klein Bottle.
   (b) Use that CW decomposition to compute the homology groups in all degrees of the Klein Bottle.
   (c) Now consider the Klein Bottle as the union of two Moebius bands with the boundary S^1 of one Moebius band identified with the boundary S^1 of the other Moebius band. Using this decomposition compute the fundamental group of the Klein Bottle using Van Kampen's Theorem.
   (d) Use the same decomposition to compute all the homology groups of the Klein Bottle using the Mayer-Vietoris Theorem.

2. Justify your answers to the questions below, meaning draw or describe the cover if there is one and prove that the cover is impossible if it is impossible.
   (a) Does the connected sum of 2 tori cover the connected sum of 3 projective planes?
   (b) Does the connected sum of 5 tori cover the connected sum of 3 tori?
   (c) Does the connected sum of 5 tori cover the connected sum of 4 tori?
   (d) Does the connected sum of 6 projective planes cover the connected sum of 2 tori?

3. The Whitehead link is a solid torus T with a simple closed curve J in it that links itself as shown.
   (a) Draw the universal cover of this object.
   (b) Does J lift to a closed curve? Why or why not?
   (c) Draw all lifts of J.
   (d) Does J bound a disk in T? Why or why not?



Part 2

4. Consider the real projective plane $\mathbb{RP}^2$ with homogeneous coordinates x, y, z.

   (a) Prove that each of the equations

$$X_1 : \quad x^2 - y^2 + z^2 = 0$$
$$X_2 : \quad x^2 - 2y^2 + 3z^2 = 0$$

   defines a submanifold X_i. What is $\dim X_i$? What familiar manifold is diffeomorphic to X_i?

   (b) Show that X_1 intersects X_2 transversely.

   (c) Compute the mod 2 intersection number $I_2(X_1, X_2)$. Explain your argument carefully.

   (d) Repeat parts (a)–(c) with the linear equations

$$Y_1 : \quad x - y + z = 0$$
$$Y_2 : \quad x - 2y + 3z = 0$$

   (e) Which of X_1, X_2, Y_1, Y_2, if any, bound a compact submanifold of $\mathbb{RP}^2$?

5. (a) Let $S \subset \mathbb{A}^3$ be a compact oriented surface of genus 2. (Recall that $\mathbb{A}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ is the standard 3-dimensional affine space.) Sketch an explicit construction of a differential form $\omega \in \Omega^2(S)$ such that

$$\int_S \omega = 2005.$$

   (b) Let $i \colon S \hookrightarrow \mathbb{A}^3$ be the inclusion. Does there exist $\eta \in \Omega^2(\mathbb{A}^3)$ such that $d\eta = 0$ and $\iota^*\eta = \omega$?

6. The following three parts are not related to each other.

   (a) Does there exist a degree two map $f \colon \mathbb{RP}^3 \to S^3$? Prove nonexistence or sketch a construction.

   (b) Sketch a proof that there does not exist a nonvanishing vector field ξ on S^2.

   (c) Suppose $X^2, Y^5 \subset S^3 \times S^4$ are submanifolds of the indicated dimensions which intersect transversely. Prove that there is an even number of intersection points.