

Topology Prelim Exam, January 2005

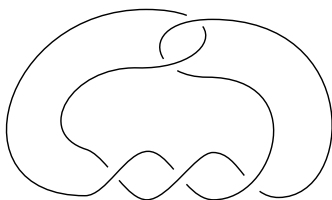
Do 2 problems from each part.

Part A: Algebraic topology

1) Let  $X$  be the wedge of the three spaces  $\mathbb{R}P^2$ ,  $S^2$ , and  $S^3$ , and let  $\tilde{X}$  be the universal cover of  $X$ . Compute the homology groups  $H_i(\tilde{X})$  for all  $i$ .

2 a) Let  $S$  be a compact solid torus. Compute all the homology groups of  $S$  and compute all the homology groups of  $\partial S$ .

b) Let  $T$  be the (long and thin) knotted solid torus in  $\mathbb{R}^3$  shown below, and let  $S^3$  be the one-point compactification of  $\mathbb{R}^3$ . Compute all the homology groups of  $\overline{S^3 - T}$ .



3) A surface  $F_1$  is a  $d$ -fold branched cover of a surface  $F_2$  if we can remove finitely many disks from  $F_1$ , and finitely many disks from  $F_2$ , to obtain surfaces-with-boundary  $\hat{F}_1$  and  $\hat{F}_2$ , with a  $d$ -fold covering map  $\hat{F}_1 \rightarrow \hat{F}_2$ .

a) Is  $S^2$  a branched cover of  $T^2$ ? If so, what are the possibilities for  $d$ ?

b) Is  $T^2$  a 2-fold branched cover of  $S^2$ ? If so, how many disks are removed? (If there is more than one way to do the cover, one answer is enough.)

Part B: Differential topology

4) Let  $S^1 \subset \mathbb{C}$  be the unit circle centered at the origin, and let  $i : T^2 = S^1 \times S^1 \rightarrow \mathbb{C} \times \mathbb{C}$  be inclusion.

a) If  $\pi : \mathbb{C}^2 - \{0, 0\} \rightarrow \mathbb{C}\mathbb{P}^1$  is the usual projection map, compute the degree of  $\pi \circ i : T^2 \rightarrow \mathbb{C}\mathbb{P}^1$ .

b) Let  $\rho : T^2 \rightarrow T^2$  be obtained by restricting complex conjugation on  $\mathbb{C}^2$ . Compute the degree of  $\rho$  and the Lefschetz number  $L(\rho)$ .

c) Determine whether  $\pi \circ i$  and/or  $\rho$  are null homotopic.

5. Consider the map  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $f(x, y, z) = r^2 \cdot (x, y, z) + g(x, y, z)$ , where  $g(x, y, z) = (y \ln(x^2 y^2 z^2 + 7), z^2 \ln(\ln(e^x + 10)), xy \cos(z) + xz \cos(y) + \sqrt{37})$  and  $r^2 = x^2 + y^2 + z^2$ .

a) Show that the equation  $f(x, y, z) = 0$  has at least one solution.

b) Show that  $f$  is onto.

6. On  $\mathbb{R}^3$ , let

$$\omega = xdy \wedge dz + ydx \wedge dz + zdx \wedge dy + \cos(x)dx \wedge dy + e^y dy \wedge dz + z^2 dx \wedge dz.$$

a) Is  $\omega$  closed? If not, compute  $d\omega$ . Is  $\omega$  exact? If so, find a 1-form  $\alpha$  such that  $d\alpha = \omega$ .

b) Let  $\nu$  be the restriction of  $\omega$  to the unit sphere. As a form on  $S^2$ , is  $\nu$  closed? Is  $\nu$  exact?

c) Compute  $\int_{S^2} \nu$ .