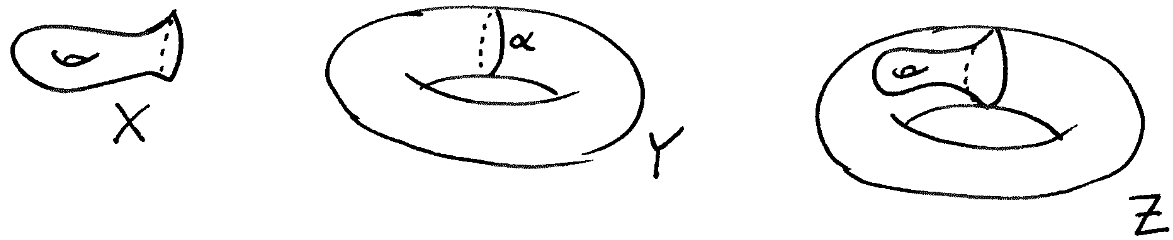
Topology Prelim

January 11, 2006

Directions: You have four hours. Answer two questions from part A and two questions from part B. Specify clearly which problems you want graded.

Part A

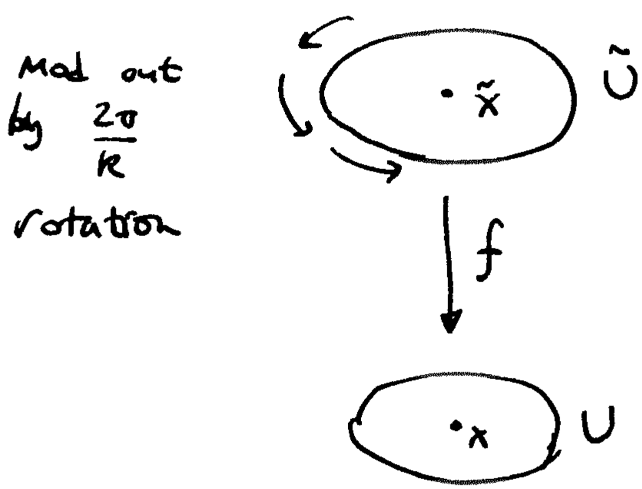
A1. Let X be the torus, minus the interior of a closed disk, and let Y be a torus. Let Z be the space obtained by identifying the boundary circle of X with the loop marked α in the picture:



(a) Give a presentation for the fundamental group of Z. (You may take the fundamental group of Y as known.)
(b) Compute $H_*(Z)$. (You may take the homology of Y as known.)

A2. Suppose $f : X \to Y$ is a 2-sheeted regular covering space, with X connected. Prove that the induced map $f_* : H_1(X) \to H_1(Y)$ is not surjective. (You may use the fact that H_1 is the abelianization of π_1.)

A3. Let D be the open unit disk in the complex plane $\mathbf{C}$. A n-sheeted *branched covering space* of a closed surface S is a map $f : \tilde{S} \to S$, where $\tilde{S}$ is a surface and the following properties hold: (1) There is a finite subset X of S such that the restriction of f to $f^{-1}(S - X) \to S - X$ is an n-sheeted covering space. (2) For each $x \in X$ there exists an open disk U containing x such that $f^{-1}(U)$ is the disjoint union of open disks, and if $\tilde{U}$ is any one of these disks, then there are homeomorphisms $\tilde{g} : \tilde{U} \to D$ and $g : U \to D$, such that $g(x) = 0$ and $g \circ f \circ \tilde{g}^{-1} : D \to D$ is the map $z \mapsto z^k$. The following picture is meant to give the idea of the map $\tilde{U} \to U$:



In the setting of (2), we write $\tilde{x}$ for the unique point of $\tilde{U}$ lying over x, and we say that f has k-fold branching at $\tilde{x}$.

Now, suppose S is the sphere S^2 and $\tilde{S}$ is a 168-sheeted branched covering space of S, with X consisting of three points x, y and z, and f having 2-fold (resp. 3-fold, 7-fold) branching at each point of $\tilde{S}$ lying over x (resp. y, z). Compute the genus of $\tilde{S}$.

Part B

B1. Let $f : \mathbf{R}^5 \to \mathbf{R}^3$ be a smooth map. Show that there exists a sphere S in $\mathbf{R}^3$, centered at the origin, such that $f^{-1}(S)$ is a smooth (but possibly empty) 4-dimensional submanifold of $\mathbf{R}^5$.

B2. Let $\mathbf{R}^2$ be parametrized by (u, v), and let $\mathbf{R}^3$ be parametrized by (x, y, z). In $\mathbf{R}^2$, let $U = (0, \pi) \times (0, 2\pi)$, and let $f : \mathbf{R}^2 \to \mathbf{R}^3$ be given by

$$f(u, v) = \big(\sin(u)\cos(v), \sin(u)\sin(v), \cos(u)\big) .$$

(a) Compute $f^*(x)$, $f^*(y)$, $f^*(z)$, $f^*(dx)$, $f^*(dy)$ and $f^*(dz)$.

(b) Consider the differential form $\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$ on $\mathbf{R}^3$. Compute $\int_U f^*(\omega)$.

B3. Consider the 2-sphere S^2 as the quotient of $\mathbf{R^3} - \{(0, 0, 0)\}$ under the equivalence relation $(x_1, x_2, x_3) \sim (tx_1, tx_2, tx_3)$, $t > 0$. Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be $f(x, y, z) = (x^2, y^2, z^2)$. Then f induces $\bar{f} : S^2 \to S^2$.

(a) Find two Lefschetz fixed points for $\bar{f}$ with different local Lefschetz numbers.

(b) Use the fact that $\bar{f}$ is not onto to show that it is homotopic to a constant map. Then use this to compute the Lefschetz number of $\bar{f}$.