OPTIMAL PORTFOLIO-CONSUMPTION WITH HABIT FORMATION AND PARTIAL OBSERVATIONS: THE FULLY EXPLICIT SOLUTIONS APPROACH

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ABSTRACT. We consider a model of optimal investment and consumption with both habit-formation and partial observations in incomplete Itô processes markets. The individual investor develops addictive consumption habits gradually while he can only observe the market stock prices but not the instantaneous rates of return, which follow Ornstein-Uhlenbeck processes. Applying the Kalman-Bucy filtering theorem and Dynamic Programming arguments, we solve the associated HJB equation fully explicitly for this path dependent stochastic control problem in the case of power utility preferences. We will provide the optimal investment and consumption policies in explicit feedback forms using rigorous verification arguments.

1. Introduction

Habit Formation preference has become a popular alternative tool of traditional time separable von Neumann-Morgenstern utilities on consumption optimization problems during recent years. It has been observed that the time additivity property of traditional preferences is not consistent with many empirical experiments, for instance, the celebrated Equity Premium Puzzle. (see Mehra and Prescott [12] and Constantinides [4]). The literature in financial economics has been arguing the past consumption pattern has the impact on individual’s current consumption decisions, and the preference should depend both on the consumption rate process and the corresponding consumption history integral. In particular, the path dependent linear habit formation preference \( \mathbb{E}[\int_0^T U(t, c_t - Z_t) dt] \) has been widely accepted, where index \( Z_t \) stands for the accumulative consumption history. The definition that instantaneous utility function is decreasing in \( Z_t \) indicates that an increase in consumption today increases current utility but depresses all future utilities through the induced increase in future standards of living.

Various works studied the continuous time habit-formation optimization problem in complete Itô processes markets under full information. Detemple and Zapatero [5] and [6] employed the martingale approach and derived the closed form optimal consumption by establishing some recursive stochastic differential equations. In the same framework, Schroder and Skiadas [16] made an insightful observation that under some appropriate assumptions, to solve the optimal portfolio selection with utilities incorporating linear habit formation \( \mathbb{E}[\int_0^T U(t, c_t - Z_t) dt] \) is equivalent to solve the time additive utility maximization \( \mathbb{E}[\int_0^T U(t, c'_t) dt] \) in some isomorphic financial markets without habit formation, where the optimal policies are related as \( c'_t = c_t^* - Z_t^* \). Munk [13] adopts the
Market Isomorphism result in his complete market model with stochastic interests rates processes, and provided closed form optimal strategies in some special cases. Taking advantage of the first order condition in the same complete market, possibly non-Markovian, Egglezos and Karatzas \cite{7} exploited the interplay of stochastic partial differential equations to the utility maximization with linear addictive habit formation, and obtained some stochastic feedback formulae for the optimal portfolio and consumption choices. Although many nice results have already been explored, the problem remains open when the financial market turns into incomplete.

In our present work, we are considering the habit forming investor in incomplete Itô processes financial markets, together with additional partial observations constraint. We are facing the case that the individual investor develops his own consumption habits during the whole investment process and meanwhile has only access to the stock price information $\mathbb{F}^S$. In other words, he can not observe the mean rate of return process $\mu_t$ and the corresponding Brownian motion $W_t$ which appeared in the stock price dynamics. In our model, we will assume $\mu_t$ follows the mean reverting Ornstein Uhlenbeck process driven by another Brownian motion $B_t$.

Optimal investment problems under incomplete information have been studied by numerous authors, and we only list a very small amount of them: Lakner \cite{11} applied martingale methods and derived the structure of the optimal investment strategies. The linear diffusion model is studied by Brendle \cite{2}, who derived explicit results for the value of information on optimal investment with power and exponential utilities using dynamic programming approach. The effects of learning on the composition of the optimal portfolios are studied in Brennan \cite{3} and Xia \cite{17}. Björk, Davis and Landén \cite{1} considered the market model with unobservable rates of returns that are allowed to be arbitrary semimartingales, and they provided a unified treatment of a large class of partially observed investment problems and explicit representation of the optimal terminal wealth and portfolio strategies.

Our contributions are two folds. From the modeling perspective, we are considering the utility maximization problem with consumption habit formation in the incomplete financial markets setting, where there exist Brownian motions $W_t$ and $B_t$ that are not perfectly correlated. Together, our individual investor withstands the incomplete information restriction, that he only has access to the public stock prices. The combination of these two scenarios is not only a novel framework and mathematically interesting, but also it covers many economically realistic constraints that the individual investor is facing in his daily lives. On the other hand, mathematically speaking, we solved the relatively complicated nonlinear HJB equation in a fully explicit form, and derived the $\mathcal{F}_t^S$-adapted optimal investment and consumption polices in feedback form via rigorous verification arguments.

The structure of the present paper is as follows: Section 2 introduces the market model and the concept of habit formation process. Section 3 briefly reviewed the stochastic filtering result and Kalman Bucy filtering Theorem. The utility maximization problem with addictive habit formation and partial observations is defined in Section 4. By applying the Kalman Bucy filtering theorem and Dynamic Programming Principle, we formally derived the Hamilton-Jacobi-Bellman(HJB) equation for the power utility preference, and we provide the decoupled form solution of this nonlinear PDE, which reduces the algorithm to solving some auxiliary ODEs with constant coefficients. Based
on these classical solutions, the explicit feedback form of the optimal investment and consumption policies will be obtained. Section 5 contains the rigorous proofs of the verification arguments. At last, four cases of fully explicit solutions of some auxiliary ODEs are presented in the Appendix A.

2. Market Model and Consumption Habit Formation

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, we consider a financial market with one riskfree bond and one stock account for a “small investor” in a finite time horizon $[0, T]$. The price of the bond $S^0_t$ solves:

$$dS^0_t = r_t S^0_t dt, \quad 0 \leq t \leq T$$

with initial price $S^0_0 = 1$, and without loss of generality, we assume the interest rate $r_t \equiv 0$, for all $t \in [0, T]$, this can be achieved by the standard change of numéraire.

The stock price $S_t$ is modeled as a diffusion process solving:

$$dS_t = \mu_t S_t dt + \sigma S_t dW_t, \quad 0 \leq t \leq T,$$

with $S_0 = s > 0$, where the drift process $\mu_t$ is $\mathcal{F}_t$-progressively measurable, and satisfying the mean-reverting Ornstein Uhlenbeck SDE:

$$d\mu_t = -\lambda (\mu_t - \bar{\mu}) dt + \sigma \mu dB_t, \quad 0 \leq t \leq T,$$

$W_t$ and $B_t$ are Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and they are correlated with the coefficient $\rho \in [-1, 1]$. We assume the initial value of the drift process $\mu_0$ is a $\mathcal{F}_0$ measurable Gaussian random variable, satisfying $\mu_0 \sim N(\eta, \theta)$, which is independent of Brownian motions $(B_t)_{0 \leq t \leq T}$ and $(W_t)_{0 \leq t \leq T}$. We also assume all the coefficients $\sigma S, \lambda, \bar{\mu}, \sigma \mu \geq 0$ are nonnegative constants.

Now, at each time $t \in [0, T]$, the investor chooses a consumption rate $c_t \geq 0$, and decides the amounts $\pi_t$ of her wealth to invest in the stock, and the rest in bank. Then, in this self-financing market model, the investor’s total wealth process $X_t$ will follow the dynamics:

$$dX_t = (\pi_t \mu_t - c_t) ds + \sigma_S \pi_t dW_t, \quad 0 \leq t \leq T,$$

with the initial wealth $X_0 = x > 0$.

In this paper, we adopt the notation $Z_t = Z(t; c_t)$ as “Habit Formation” process or “the standard of living” process to describe his “consumption habits level”. We assume the accumulative index $Z_t$ follows the dynamics:

$$dZ_t = (\delta(t)c_t - \alpha(t)Z_t) dt, \quad 0 \leq t \leq T,$$

where $Z_0 = z \geq 0$ is called the initial habit, and $\alpha(t), \delta(t)$ are nonnegative continuous functions.

Equivalently, (2.4) stipulates

$$Z_t = ze^{-\int_0^t \alpha(u) du} + \int_0^t \delta(u)e^{-\int_u^t \alpha(v) dv} \pi_u du, \quad 0 \leq t \leq T,$$

$Z_t$ is defined as an exponentially weighted average of the initial habit and the past consumption. Here, functions $\alpha(t)$ and $\delta(t)$ measure, respectively, the persistence of the past level and the intensity
of consumption history.

In our current work, we are considering the case of "addictive habits", i.e., we require investor’s current consumption strategies shall never fall below the standard of living level,

\[ c_t \geq Z_t, \quad \forall 0 \leq t \leq T, \quad \text{a.s..} \]

We will see in the future that this additional consumption budget constraint implies initial wealth must be sufficiently large to sustain habits and ensure the existence of optimal policies.

3. Stochastic filtering with partial observations

From now on, we shall make the assumption that the investor can observe the stock price process \( S_t \) which are published and available to the public, however, the drift process \( \mu_t \) and the information of Brownian motions \( (W_t)_{0 \leq t \leq T} \) and \( (B_t)_{0 \leq t \leq T} \) are unknown. We shall call this as the "partial observations information" scenario. This investment and consumption optimization problem with incomplete information will be modeled by requiring the investment strategy \( (\pi_t)_{0 \leq t \leq T} \) and consumption policy \( (c_t)_{0 \leq t \leq T} \) must be only adapted to the partial observation filtration \( \mathcal{F}_t^S = (\mathcal{F}_t^S)_{0 \leq t \leq T} \) where \( \mathcal{F}_t^S = \sigma\{S_u : 0 \leq u \leq t\} \), which is strictly smaller than the background full information \( \mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \). To this end, we need first to prepare the stochastic filtering theorem.

3.1. General filtering model. Consider a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) equipped with a background filtration \( \mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \). All processes are assumed to be \( \mathcal{F} \) progressively measurable. The stochastic signal state process \( R_t : \Omega \rightarrow \mathbb{R}^n \) is given by the SDE

\[ dR_t = A(t, R_t)dt + C(t, R_t)dB_t, \quad 0 \leq t \leq T, \]

where \( B_t \) denotes a \( \mathcal{F}_t \)-Brownian Motion. We assume \( R_t \) is not directly observable.

And an observation process \( H_t \) is given as:

\[ dH_t = D(t, R_t)dt + E(t, R_t)dW_t, \quad 0 \leq t \leq T, \]

where, \( W_t \) is another \( \mathcal{F}_t \)-Brownian motion, correlated with \( B_t \) with the coefficient \( \rho \in [-1, 1] \).

The filtering problem is to find the best estimate \( \hat{R}_t \) based on the observation of \( H_t \): which means to compute the conditional distribution of the signal process \( \hat{R}_t \), given the observation filtration \( \mathcal{F}_t = \sigma(H_s : 0 \leq s \leq t) \):

\[ \mathbb{P}\left[R_t \in A \left| \mathcal{F}_t \right.\right], \quad 0 \leq t \leq T, \quad \text{for } A \in \mathcal{F}_T, \]

equivalently, to compute the conditional expectation:

\[ E\left[f(R_t) \left| \mathcal{F}_t \right.\right], \quad 0 \leq t \leq T, \]

for a suitable class of test functions \( f : \mathbb{R} \rightarrow \mathbb{R} \).
3.2. The Kalman Bucy filtering. For simplicity, we look at the one dimensional signal state process $R_t$ and observation process $H_t$, defined by linear SDEs:

\[
\begin{align*}
\frac{dR_t}{dt} &= A(t)R_t dt + C(t)dB_t \\
\frac{dH_t}{dt} &= D(t)R_t dt + E(t)dW_t, \quad 0 \leq t \leq T,
\end{align*}
\]

where, $A(t)$, $C(t) \neq 0$, $D(t)$, $E(t) \neq 0$ are deterministic functions satisfying the integrability condition:

\[
\int_0^T \left[ |A(t)| + |D(t)| + C^2(t) + E^2(t) \right] dt < +\infty.
\]

Suppose $R_0$ is a Gaussian random variable with $R_0 \sim N(\mu, \theta)$, which is independent of Brownian motions $(B_t)_{0 \leq t \leq T}$ and $(W_t)_{0 \leq t \leq T}$. And $H_0 = s$, where $s$ is a constant.

One can see $(R_t, H_t)$ is a Gaussian vector process, so the conditional distribution of $R_t$ given $\hat{F}_t = \sigma(H_s : 0 \leq s \leq t)$ is also Gaussian, and the entire distribution is determined by its mean $\hat{R}_t$ and variance $\hat{\Omega}_t$:

\[
\begin{align*}
\hat{R}_t &= \mathbb{E} \left[ R_t \mid \hat{F}_t \right], \\
\hat{\Omega}_t &= \mathbb{E} \left[ (R_t - \hat{R}_t)^2 \mid \hat{F}_t \right],
\end{align*}
\]

with initial values

\[
\begin{align*}
\hat{R}_0 &= \mathbb{E} \left[ R_0 \mid \hat{F}_0 \right] = \mathbb{E} \left[ R_0 \right] = \mu \\
\hat{\Omega}_0 &= \mathbb{E} \left[ (R_0 - \hat{R}_0)^2 \mid \hat{F}_0 \right] = \mathbb{E} \left[ (R_0 - \mu)^2 \right] = \text{Var} \left[ R_0 \right] = \theta.
\end{align*}
\]

And the linear filtering problem becomes to compute the $\hat{R}_t$ and $\hat{\Omega}_t$ given their initial values. The celebrated Kalman-Bucy filter theorem gives us the algorithm for these estimations.

**Theorem 3.1** (One-dimensional Kalman-Bucy Filtering).

For the one dimensional signal state process $R_t$ and observation process $H_t$ following the linear SDEs (3.1), the conditional expectation $\hat{R}_t = \mathbb{E} \left[ R_t \mid \hat{F}_t \right]$ satisfies the SDE:

\[
\frac{d\hat{R}_t}{dt} = A(t)\hat{R}_t dt + \left[ \frac{D(t)}{E(t)} \hat{\Omega}_t + C(t)\rho \right] d\hat{W}_t, \quad 0 \leq t \leq T,
\]

with $\hat{R}_0 = \mu$, where $(\hat{W}_t)_{0 \leq t \leq T}$ is called the **Innovation Process**, and defined as:

\[
\frac{d\hat{W}_t}{dt} = \frac{D(t)}{E(t)} \left( R_t - \hat{R}_t \right) dt, \quad 0 \leq t \leq T, \quad \hat{W}_0 = 0.
\]

and $\hat{W}_t$ is an $\hat{F}_t$ progressively measurable Brownian Motion.

The conditional variance $\hat{\Omega}_t = \text{Var} \left[ R_t \mid \hat{F}_t \right]$ satisfies the deterministic Ricatti equation:

\[
\frac{d\hat{\Omega}_t}{dt} = - \frac{D^2(t)}{E^2(t)} \hat{\Omega}_t^2 + 2 \left[ A(t) - \frac{C(t)D(t)\rho}{E(t)} \right] \hat{\Omega}_t + C^2(t)(1 - \rho^2), \quad 0 \leq t \leq T,
\]

with the initial condition $\hat{\Omega}_0 = \theta$. 
4. Utility Maximization with Kalman-Bucy Filtering


Apply the above Kalman-Bucy Stochastic Filtering Theorem, we can first define the Innovation Process in our market model as:

$$
\frac{d\hat{W}_t}{\sigma_S} = \frac{1}{\sigma_S} \left( (\mu_t - \bar{\mu})dt + \sigma_S dW_t \right), \quad 0 \leq t \leq T,
$$

which is a Brownian motion under partial observations filtration $\mathcal{F}^S_t$.

Moreover, by Kalman-Bucy filtering theorem, the conditional estimations of drift process: $\hat{\mu}_t = \mathbb{E} \left[ \mu_t | \mathcal{F}_S^S \right]$ satisfies the SDE:

$$
d\hat{\mu}_t = -\lambda (\hat{\mu}_t - \bar{\mu}) dt + \left( \frac{\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho}{\sigma_S^2} \right) d\hat{W}_t, 
$$

with $\hat{\mu}_0 = \mathbb{E} \left[ \mu_0 | \mathcal{F}_0^S \right] = \eta$. So we can solve $\hat{\mu}_t$ as the strong solution of the SDE [4.2] by knowing the stock price process $S_t$.

And the conditional variance $\hat{\Omega}_t = \mathbb{E} \left[ (\mu_t - \hat{\mu}_t)^2 | \mathcal{F}_S^S \right]$ satisfies the Riccati ODE:

$$
d\hat{\Omega}_t = \left[ \sigma_\mu^2 - 2 \lambda \hat{\Omega}_t - \frac{(\hat{\Omega}_t + \sigma_S \sigma_\mu \rho)^2}{\sigma_S^2} \right] dt, \quad 0 \leq t \leq T,
$$

with $\hat{\Omega}(0) = \mathbb{E} \left[ (\mu_0 - \eta)^2 | \mathcal{F}_0^S \right] = \theta$, which has an explicit solution as:

$$
\hat{\Omega}_t = \hat{\Omega}(t; \theta) = \sqrt{k} \sigma_S \frac{k_1 \exp(2(\frac{\sqrt{\theta}}{\sigma_S}) t) + k_2}{k_1 \exp(2(\frac{\sqrt{\theta}}{\sigma_S}) t) - k_2} - \left( \lambda + \frac{\sigma_\mu \rho}{\sigma_S} \right) \sigma_S^2, \quad 0 \leq t \leq T,
$$

where:

$$
k = \lambda^2 \sigma_S^2 S + 2 \sigma_S \sigma_\mu \lambda \rho + \sigma_\mu^2,
\quad k_1 = \sqrt{k} \sigma_S + (\lambda^2 \sigma_S^2 + \sigma_S \sigma_\mu \rho) + \theta,
\quad k_2 = -\sqrt{k} \sigma_S + (\lambda^2 \sigma_S^2 + \sigma_S \sigma_\mu \rho) + \theta.
$$

By observation, we see $\hat{\Omega}(t)$ tends to the value

$$
\theta^* = \sigma_S \sqrt{\lambda^2 \sigma_S^2 S + 2 \sigma_S \sigma_\mu \lambda \rho + \sigma_\mu^2 - (\lambda^2 \sigma_S^2 + \sigma_S \sigma_\mu \rho)},
$$

as time $t \to +\infty$, which we call as “steady state learning” (see also Brennan [3]). This convergence property of $\hat{\Omega}(t)$ tells us the precision of the drift estimate goes from an initial condition to a steady state in the long time run, and after large time $T$, new return observations contribute to updating the estimated value of the state variable, but seldom reduce the variance of the estimation error.

Moreover, by the Riccati ODE [4.3], we clearly have the boundedness

$$
\min(\theta, \theta^*) \leq \hat{\Omega}(t) \leq \max(\theta, \theta^*), \quad t \in [0, T].
$$
Under the observation filtration \((F^S_t)_{0 \leq t \leq T}\), we can instead rewrite stock price dynamics (2.1) driven by the innovation process \(\hat{W}_t\) as:

\[
dS_t = \hat{\mu}_t S_t dt + \sigma S_t d\hat{W}_t, \quad 0 \leq t \leq T.
\]

Notice we are now seeking the optimal investment and consumption strategies \(\pi_t\) and \(c_t\) which are only progressively measurable with respect to the partial observations filtration \(F^S_t\), where the living standard process \(Z_t\) satisfies the ODE

\[
dZ_t = (\delta (t) c_t - \alpha (t) Z_t) dt, \quad 0 \leq t \leq T,
\]

and under the partial observations filtration \(F^S_t\), the wealth process dynamics (2.3) under \(\pi_t\) and \(c_t\) will be rewritten as:

\[
dx_t = (\pi_t \hat{\mu}_t - c_t) dt + \sigma \pi_t d\hat{W}_t, \quad 0 \leq t \leq T.
\]

Our goal now is to maximize the consumption with linear habit formation and terminal wealth by power utility preference under the partial observations filtration \(F^S_t\):

\[
V(t, x, z, \eta; \theta) = \sup_{\pi, c \in A} \mathbb{E} \left[ \int_t^T \frac{(c_s - Z_s)^p}{p} ds + \frac{(X_T)^p}{p} \left| X_t = x, Z_t = z, \hat{\mu}_t = \eta, \hat{\Omega}_t = \theta \right. \right],
\]

where we take the risk aversion coefficient \(p < 1\) and \(p \neq 0\).

Recall that the conditional variance \(\Omega_t\) is a deterministic function of time, we can set \(\hat{\Omega}_t = \theta\) as the parameter instead of the variable, and the dimension of the value function can be reduced as:

\[
V(t, x, z, \eta; \theta) = \sup_{\pi, c \in A} \mathbb{E} \left[ \int_t^T \frac{(c_s - Z_s)^p}{p} ds + \frac{(X_T)^p}{p} \left| X_t = x, Z_t = z, \hat{\mu}_t = \eta, \hat{\Omega}_t = \theta \right. \right];
\]

for each fixed initial value \(\hat{\Omega}(t) = \theta\).

An investment and consumption pair process \((\pi_t, c_t)\) is said in the Admissible Control Space \(A\): if it is \(F^S_t\)-progressively measurable, and satisfies the integrability conditions:

\[
\int_0^T \pi_t^2 dt < +\infty, \quad a.s. \quad \text{and} \quad \int_0^T c_t dt < +\infty, \quad a.s.
\]

with the addictive habits constraint that: \(c_t \geq Z_t, \quad \forall t \in [0, T]\). Moreover, no bankruptcy is allowed, i.e., the investor’s wealth remains nonnegative: \(X_t \geq 0, \quad 0 \leq t \leq T\).

By the definition of \(V(t, x, z, \eta)\) and Dynamic Programming Principle, we can formally derive the Hamilton-Jacobi-Bellman (HJB) equation for the value function as:

\[
V_t - \alpha (t) z V_z - \lambda (\eta - \bar{\mu}) V_\eta + \frac{(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho)^2}{2 \sigma_S^2} V_{\eta \eta} + \max_c \left[-c V_x + c \delta (t) V_x \right. \\
+ \frac{(c - z)^p}{p} \] + \max \pi \left[ \pi \eta V_x + \frac{1}{2} \sigma_S^2 \pi^2 V_{xx} + V_{x \eta} \left( \hat{\Omega}(t) + \sigma_S \sigma_\mu \rho \right) \pi \right] = 0,
\]

\[
(4.13)
\]
with the terminal condition \( V(T, x, z, \eta) = \frac{z^p}{p} \).

### 4.2. Decoupled Reduced Form Solutions.

If \( V(t, x, z, \eta) \) is smooth enough, the first order condition formally derives

\[
\pi^*(t, x, z, \eta) = \frac{-\eta V_x - (\hat{\Omega}(t) + \sigma S \sigma \mu \rho) V_{x\eta}}{\sigma_S^2 V_{xx}},
\]

(4.14)

\[
c^*(t, x, z, \eta) = z + \left( V_x - \delta(t) V_z \right)^{\frac{1}{p-1}}.
\]

which achieve the maximum over control policies \( \pi \) and \( c^* \) respectively.

Plugging forms of (4.14) for \( \pi^* \) and \( c^* \), the HJB equation becomes:

\[
V_t - \alpha(t)z V_x - \lambda(\eta - \bar{\mu})V_{\eta} + \frac{\left( \hat{\Omega}(t) + \sigma S \sigma \mu \rho \right)^2}{2\sigma_S^2} V_{\eta\eta} - \frac{\eta \left( \hat{\Omega}(t) + \sigma S \sigma \mu \rho \right)}{\sigma_S^2} V_x V_{x\eta} V_{xx}
\]

(4.15)

\[
- \frac{\eta^2}{2\sigma_S^2} V_{xx} - \frac{\left( \hat{\Omega}(t) + \sigma S \sigma \mu \rho \right)^2}{2\sigma_S^2} V_{x\eta} - \frac{z}{2} \left( V_x - \delta(t) V_z \right) = \frac{p-1}{p} \left( V_x - \delta(t) V_z \right)^{\frac{p}{p-1}} = 0.
\]

From definition (4.11) of the value function, and dynamics (4.9) and (4.8) for \( X_t \) and \( Z_t \) respectively, it’s easy to show that if \( V(t, x, z, \eta) \) is finite then it is homogeneous in \((x, z)\) with degree \( p \), i.e., for any \( x > 0, z \geq 0 \) and the positive constant \( k \), we have \( V(t, kx, kz, \eta) = k^p V(t, x, z, \eta) \). It therefore makes sense for us to seek the value function of the form:

\[
V(t, x, z, \eta) = \left[ \frac{(x - w(t, \eta)z)^p}{p} \right] M(t, \eta)
\]

(4.16)

for some test functions \( w(t, \eta) \) and \( M(t, \eta) \) to be determined. By the virtue of \( V(T) = \frac{z^p}{p} \), we will require \( M(T, \eta) = 1 \) and \( w(T, \eta) = 0 \).

After we do the direct substitution in the above Equation (4.15) and divide the equation on both sides by \((x + w(t, \eta)z)^p\), the HJB equation becomes

(4.17)

\[
\left[ - w_t z + \alpha(t) w z - (1 + \delta(t) w) z + \lambda(\eta - \bar{\mu}) w_\eta - \frac{\left( \hat{\Omega}(t) + \sigma S \sigma \mu \rho \right)^2}{2\sigma_S^2} w_\eta \eta - \frac{\eta \left( \hat{\Omega}(t) + \sigma S \sigma \mu \rho \right)}{\sigma_S^2} M \right] M
\]

\[
+ \frac{1}{p} M_t - \lambda(\eta - \bar{\mu}) M + \frac{\left( \hat{\Omega}(t) + \sigma S \sigma \mu \rho \right)^2}{2\sigma_S^2} M_\eta \eta - \frac{\eta^2}{2(p-1)\sigma_S^2} M - \frac{\eta \left( \hat{\Omega}(t) + \sigma S \sigma \mu \rho \right)}{(p-1)\sigma_S^2} M_\eta
\]

\[
- \frac{\left( \hat{\Omega}(t) + \sigma S \sigma \mu \rho \right)^2}{2(p-1)\sigma_S^2} M_\eta^2 M - \frac{p-1}{p} \left( 1 + \delta(t) w(t, \eta) \right)^{\frac{p}{p-1}} M_\eta^p = 0.
\]

Since the Equation (4.17) holds for all values of \( x \) and \( z \), we naturally set the unknown priori function \( w(t, \eta) = w(t) \) as a deterministic function in time \( t \) and satisfies:

(4.18)

\[- w_t(t) + \alpha(t) w(t) - (1 + \delta(t) w(t)) = 0\]
with the terminal condition \( w(T) = 0 \), which is equivalent to:

\[
(4.19) \quad w(t) = \int_t^T \exp \left( \int_t^s (\delta(v) - \alpha(v))dv \right) ds. \quad 0 \leq t \leq T.
\]

Now we can substitute the function \( w(t) \) into the equation \((4.17)\) above, and simplify it as:

\[
(4.20) \quad M_t + \frac{pn^2}{2(1 - p)\sigma_1^2} M + \frac{(\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho)^2}{2\sigma_S^2} M_{\eta \eta} + (1 - p) \left( 1 + \delta(t) w(t) \right)^{\frac{p}{p-1}} M^{\frac{p}{p-1}}
\]

\[
+ \left[ -\lambda(\eta - \tilde{\mu}) + \frac{\eta(\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho) p}{(1 - p)\sigma_S^2} \right] M + \frac{(\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho)^2}{2(1 - p)\sigma_S^2} p M_{\eta}^2 = 0.
\]

Now in order to solve the above nonlinear PDE \((4.20)\), we can set the power transform as

\[
(4.21) \quad M(t, \eta) = N(t, \eta)^{1-p}
\]

And the nonlinear PDE \((4.20)\) for \( M(t, \eta) \) reduces to the linear parabolic PDE for \( N(t, \eta) \) as:

\[
(4.22) \quad N_t + \frac{pn^2}{2(1 - p)\sigma_2^2} N(t, \eta) + \frac{(\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho)^2}{2\sigma_S^2} N_{\eta \eta} + \left( 1 + \delta(t) w(t) \right)^{\frac{p}{p-1}} N = 0
\]

with \( N(T, \eta) = 1 \).

For the above linear PDE \((4.22)\) of \( N(t, \eta) \), we can further solve it explicitly as:

\[
(4.23) \quad N(t, \eta) = \int_t^T \left( 1 + \delta(s) w(s) \right)^{\frac{p}{p-1}} \exp \left( A(s, t) \eta^2 + B(s, t) \eta + C(s, t) \right) ds
\]

\[
+ \exp \left( A(T, t) \eta^2 + B(T, t) \eta + C(T, t) \right),
\]

where we have for \( 0 \leq t \leq s \leq T \), \( A(s, t) = A(t, s) \), \( B(s, t) = B(t, s) \) and \( C(s, t) = C(t, s) \) satisfying the following ODEs:

\[
(4.24) \quad A_t(t, s) + \frac{p}{2(1 - p)\sigma_S^2} A(t, s) + \left[ -\lambda + \frac{p(\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho)}{\sigma_S^2(1 - p)} \right] A(t, s) + \frac{2(\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho)^2}{\sigma_S^2} A^2(t, s) = 0;
\]

\[
(4.25) \quad B_t(t, s) + \left[ -\lambda + \frac{p(\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho)}{\sigma_S^2(1 - p)} \right] B(t, s) + 2\lambda \tilde{\mu} A(t, s) + \frac{2(\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho)^2}{\sigma_S^2} A(t, s) B(t, s) = 0;
\]

\[
(4.26) \quad C_t(t, s) + \lambda \tilde{\mu} B(t, s) + \frac{(\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho)^2}{2\sigma_S^2} \left( B^2(t, s) + 2A(t, s) \right) = 0;
\]

with terminal conditions: \( A(s, s) = B(s, s) = C(s, s) = 0 \).
We remark that the above ODEs are similar to the ODEs obtained by Brendle for terminal wealth optimization problem with partial observations, and Brendle made an insightful observation that we can actually solve the above ODEs by solving some auxiliary ODEs with constant coefficients:

**Theorem 4.1.** For $0 \leq t \leq s \leq T$, consider the following auxiliary ODEs for $a(t; s), b(t; s), c(t; s), f(t; s)$ and $g(t; s)$:

\begin{align}
(4.27) & \quad a_t(t; s) = -\frac{2(1 - p + pp^2)}{1 - p} \sigma_a^2 a(t; s) + \left( 2\lambda - \frac{2pp\sigma_{\mu}}{(1 - p)\sigma_S} \right) a(t; s) - \frac{p}{2(1 - p)\sigma_S^2}, \\
(4.28) & \quad b_t(t; s) = -\frac{2(1 - p + pp^2)}{1 - p} \sigma_b^2 a(t; s) b(t; s) - 2\lambda \mu a(t; s) + \left( \lambda - \frac{pp\sigma_{\mu}}{(1 - p)\sigma_S} \right) b(t; s), \\
(4.29) & \quad c_t(t; s) = -\sigma_c^2 a(t; s) - \frac{(1 - p + pp^2) \sigma_c^2}{2(1 - p)} b(t; s) - \lambda \mu b(t; s), \\
(4.30) & \quad f_t(t; s) = -2(1 - p^2) \sigma_f^2 f(t; s) + \frac{2\lambda \sigma_f + \rho \sigma_{\mu}}{\sigma_S} f(t; s) + \frac{1}{2\sigma_S^2}, \\
(4.31) & \quad g_t(t; s) = \sigma_g^2 (1 - p^2) (f(t; s) - a(t; s)),
\end{align}

with the terminal conditions $a(s; s) = b(s; s) = c(s; s) = f(s; s) = g(s; s) = 0$, and if we adopt the convention $0^0 = 0$, then for the functions defined by:

\begin{align}
(4.32) & \quad \hat{A}(t; s) = \frac{a(t; s)}{(1 - p)(1 - 2a(t; s)\hat{\Omega}(t))}, \\
& \quad \hat{B}(t; s) = \frac{b(t; s)}{(1 - p)(1 - 2a(t; s)\hat{\Omega}(t))}, \\
& \quad \hat{C}(t; s) = \frac{1}{1 - p} \left[ c(t; s) + \frac{\hat{\Omega}(t)}{(1 - 2a(t; s)\hat{\Omega}(t))} b(t; s) - \frac{1 - p}{2} \log \left( 1 - 2a(t; s)\hat{\Omega}(t) \right) \right] + \frac{p}{2} \log \left( 1 - 2f(t; s)\hat{\Omega}(t) \right) - pg(t; s),
\end{align}

we have the equivalence that:

\begin{align}
(4.33) & \quad A(t; s) = \hat{A}(t; s), \quad B(t; s) = \hat{B}(t; s), \quad C(t; s) = \hat{C}(t; s), \quad 0 \leq t \leq s \leq T.
\end{align}

**Remark 4.1.** The equivalence result reveals the fact that to solve the ODEs with variable dependent coefficients is equivalent to solve the auxiliary ODEs with constant coefficients in an order, i.e., we solve the Riccati ODE first, and substitute the solution $a(t; s)$ into ODE and solve for the solution $b(t; s)$, and etc.

Actually, we can even solve out fully explicit solutions for $a(t; s), b(t; s), c(t; s), f(t; s)$ and $g(t; s)$. We list all four different cases of fully explicit solutions in the Appendix depending on the risk aversion coefficient $p$ and the market coefficients $\sigma_S, \sigma_{\mu}, \lambda$ and $\rho$. By simple substitutions,
we can therefore solve the ODEs (4.24), (4.25), (4.26) for \( A(t; s) \), \( B(t; s) \) and \( C(t; s) \) in the fully explicit expression.

**Lemma 4.1.** Suppose the risk aversion constant \( p \) and the market coefficients \( \sigma_\delta, \sigma_\mu, \lambda, p \) satisfy

\[
(4.33) \quad \frac{p}{1-p} \leq \frac{\lambda^2 \sigma_\delta^2}{(2\lambda \rho_\sigma_\mu + \sigma_\mu^2)} \leq \frac{\lambda \sigma_\delta}{\rho_\sigma_\mu},
\]

and the solution of (4.27) satisfies \( |1 - a(t; s)\tilde{\Omega}(t)| \geq \epsilon > 0 \) for a constant \( \epsilon \) on \( 0 \leq s \leq t \leq T \), then there exist uniform constants bounds \( \bar{K}_1 > 0, \bar{K}_2 > 0 \) and \( \bar{K}_3 > 0 \) such that

\[
(4.34) \quad A(t; s) \leq \bar{K}_1, \quad B(t; s) \leq \bar{K}_2, \quad C(t; s) \leq \bar{K}_3, \quad 0 \leq t \leq s \leq T.
\]

**Proof.** Under Assumption (4.33), the explicit solution \( a(t; s) \) is bounded and we observe the form of ODEs (4.28), (4.29), then \( b(t; s), c(t; s) \) are bounded on \( 0 \leq t \leq s \leq T \) if \( a(t; s) \) is bounded. Now since the ODE (4.30) is well defined independent of the risk aversion constant \( p \), and it always admits a bounded solution \( f(t; s) < 0 \) and \( |1 - f(t; s)\tilde{\Omega}(t)| > 1 > 0 \) for \( 0 \leq t \leq s \leq T \) and hence we deduce \( g(t; s) \) is bounded for \( 0 \leq t \leq s \leq T \). Combine these with the assumption \( |1 - a(t; s)\tilde{\Omega}(t)| \geq \epsilon > 0 \) on \( 0 \leq t \leq s \leq T \), we can conclude \( A(t; s), B(t; s) \) and \( C(t; s) \) are all uniformly bounded on \( 0 \leq t \leq s \leq T \) by substitution results in Theorem 4.1 \( \square \)

Now, for \( t \in [0, T] \), \( \eta \in (-\infty, +\infty) \), we can define the effective domain for the pair \((x, z)\) as:

\[
(4.35) \quad (x, z) \in \mathbb{D}_t = \{(x', z') \in (0, +\infty) \times [0, +\infty); \ x' \geq w(t)z'\}, \ 0 \leq t \leq T,
\]

and the function

\[
(4.36) \quad \bar{V}(t, x, z, \eta) = \left[ \frac{(x - w(t)z)^p}{p} \right] \left[ \frac{1}{p} \int_t^T \left( 1 + \delta(s)w(s) \right)^{-p} \exp \left( A(s, t)\eta^2 + B(s, t)\eta + C(s, t) \right) ds \right. \\
+ \exp \left( A(T, t)\eta^2 + B(T, t)\eta + C(T, t) \right) \right]^{1-p}
\]

is well defined on \([0, T] \times \mathbb{D}_t \times \mathbb{R}\) and it’s the classical solution of the HJB equation (4.13), where \( w(t) = \int_t^T \exp(\int_t^\tau (\delta(v) - \alpha(v))dv)ds \), and \( A(s, t), B(s, t), C(s, t) \) are defined in (4.24), (4.25), (4.26).

In our main result below, we want to verify the above classical solution \( \bar{V}(t, x, z, \eta) \) equals our primal value function defined in (4.11), however, the effective domain of \( \bar{V}(t, x, z, \eta) \) deduces some constraints on the optimal wealth process \( X^*_t \) and habit formation process \( Z^*_t \). In particular, when \( t = 0 \), we have to pose the initial wealth-habit budget constraint that \( x > w(0)z \).

### 4.3. The Main Result.

**Theorem 4.2** (The Verification Theorem).

Under the initial wealth-habit budget constraint \( x > w(0)z \), either if risk aversion constant

\[
(4.37) \quad p < 0;
\]
or if

\[ 0 < p < 1, \]

\[
0 < p < 1, \quad \text{together with market coefficients } \sigma_S, \sigma_\mu, \lambda, \Theta, \rho \]
satisfy the additional assumption \[ (4.33) \] and

\[
\frac{p(1 + p)}{(1 - p)^2} < \frac{\lambda^2 \sigma_S^4}{4(\Theta + \sigma_S \sigma_\mu \rho)^2},
\]

where \( \Theta \triangleq \max\{\theta, \theta^*\} \) and \( \theta^* \) is define in \[ (4.5) \], moreover, we assume the upper boundedness \( \bar{K}_1 \)
in \[ (4.34) \] of \( A(t; s) \) in Lemma \[ 1 \] satisfies

\[
8\bar{K}_1 + 1 < \frac{\lambda \sigma_S^2}{(\Theta + \sigma_S \sigma_\mu \rho)^2}.
\]

Then, define \( V(t, x, z, \eta) \) as in \[ (4.11) \] and define \( \tilde{V}(t, x, z, \eta) \) as in \[ (4.36) \], we have the equivalence:

\[
(4.41) \quad \tilde{V}(t, x, z, \eta) = V(t, x, z, \eta).
\]

And the optimal investment policy \( \pi^*_t \) and optimal consumption policy \( c^*_t \) are given in the feedback form: \( \pi^*_t = \pi^*(t, X_t^*, Z_t^*, \hat{\mu}_t) \) and \( c^*_t = c^*(t, X_t^*, Z_t^*, \hat{\mu}_t) \), \( 0 \leq t \leq T \), where the function \( \pi^*(t, x, z, \eta) : [0, T] \times \mathbb{D}_t \times \mathbb{R} \to \mathbb{R} \) is defined by:

\[
(4.42) \quad \pi^*(t, x, z, \eta) = \left[ \frac{\eta}{(1 - p)\sigma_S^2} + \frac{(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho) N_\eta(t, \eta)}{\sigma_S^2} \right] (x - w(t)z), \quad 0 \leq t \leq T.
\]

\( c^*(t, x, z, \eta) : [0, T] \times \mathbb{D}_t \times \mathbb{R} \to \mathbb{R}_+ \) is defined by:

\[
(4.43) \quad c^*(t, x, z, \eta) = z + \frac{(x - w(t)z)}{(1 + \delta(t)w(t))^{\frac{1}{1 - p}}N(t, \eta)}, \quad 0 \leq t \leq T.
\]

And the optimal wealth process \( X_t^* \), for \( 0 \leq t \leq T \), is given explicitly by:

\[
(4.44) \quad X_t^* = (x - w(0)z)\frac{N(t, \hat{\mu}_t)}{N(0, \eta)} \exp \left( \int_0^t \frac{(\hat{\mu}_u)^2}{2(1 - p)\sigma_S^2} du + \int_0^t \frac{\hat{\mu}_u}{(1 - p)} d\hat{W}_u \right) + w(t)Z_t^*,
\]

where \( w(t) \) and \( N(t, \eta) \) are defined in \[ (4.19) \] and \[ (4.23) \] respectively.

**Remark 4.2.** The more complex structure of feedback forms of optimal investment and consumption policies is the consequence of the time non-separability of the instantaneous utility with habit formation. We can see the portfolio/wealth ratio \( \frac{\pi^*}{X^*} \) and consumption/wealth ratio \( \frac{c^*}{X^*} \) are now depending on the habit-formation/wealth ratio \( \frac{Z^*}{X^*} \):

\[
\frac{\pi^*}{X^*} = \left[ \frac{\hat{\mu}_t}{(1 - p)\sigma_S^2} + \frac{(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho) N_\eta(t, \hat{\mu}_t)}{\sigma_S^2} \right] (1 - w(t)\frac{Z^*}{X^*}),
\]

and

\[
\frac{c^*}{X^*} = \left[ \frac{1}{(1 + \delta(t)w(t))^{\frac{1}{1 - p}}N(t, \hat{\mu}_t)} + \left( 1 - \frac{w(t)}{(1 + \delta(t)w(t))^{\frac{1}{1 - p}}N(t, \hat{\mu}_t)} \right) \right] \frac{Z^*}{X^*},
\]
Moreover, although function $c^*(\cdot, x, \cdot, \cdot, \cdot)$ remains still linear and increasing in $x > 0$, $c^*(\cdot, \cdot, z, \cdot)$ is not necessarily increasing in $z \geq 0$, which shows the increase of initial habit dose not necessarily imply the increase of optimal consumption stream. And since the dependence of $c^*(t, x, z, \eta)$ on the discounting factors $\alpha(t)$ and $\delta(t)$ are even more complicated, the optimal consumption process $c^*_t$ is not necessarily monotone in the habit formation process $Z^*_t$.

5. Proof of the Verification Theorem

We will first show the consumption constraint $c_t \geq Z_t$ implies the constraint on the controlled wealth process by the following proposition:

**Proposition 5.1.** The admissible space $A$ is not empty if and only if $x \geq w(0) z$. Moreover, for each pair of investment and consumption policy $(\pi, c) \in A$, the controlled wealth process $X^{\pi, c}$ satisfies the constraint:

$$X^{\pi, c}_t \geq w(t) Z_t, \quad 0 \leq t \leq T,$$

where the deterministic function $w(t)$ is defined in (4.18) and refers the cost of subsistence consumption per unit of standard of living at time $t$.

**Proof.** On one hand, let’s assume $x \geq w(0) z$, then we can always take $\pi_t \equiv 0$, and $c_t = z \exp \left( \int_0^t (\delta(v) - \alpha(v))dv \right)$ for $t \in [0, T]$, it is easy to verify $X^{\pi, c}_t \geq 0$ and $c_t \equiv Z_t$, so that $(\pi, c) \in A$, and hence $A$ is not empty.

On the other hand, starting from $t = 0$ with the wealth $x$ and the standard of living $z$, the addictive habits constraint $c_t \geq Z_t, \quad 0 \leq t \leq T$ implies the consumption must always exceed the subsistence consumption $\tilde{c}_t = Z(t; \tilde{c}_t)$ which satisfies

$$d\tilde{c}_t = (\delta(t) - \alpha(t))\tilde{c}_tdt, \quad \tilde{c}_0 = z, \quad 0 \leq t \leq T,$$

And $c_t \geq \tilde{c}_t$ is equivalent to

$$c_t \geq z \exp \left( \int_0^t (\delta(v) - \alpha(v))dv \right), \quad 0 \leq t \leq T.$$

Define the exponential local martingale

$$\tilde{H}_t = \exp \left( - \int_0^t \frac{\dot{\mu}_v}{\sigma_S} d\tilde{W}_v - \frac{1}{2} \int_0^t \frac{\dot{\mu}_v^2}{\sigma_S^2} dv \right), \quad 0 \leq t \leq T.$$

Bene’s condition implies $\tilde{H}$ is a true martingale with respect to $(\Omega, \mathcal{F}, \mathbb{P})$, since $\dot{\mu}_t$ follows the dynamics (4.2).

Now define the probability measure $\mathbb{P}$ as $d\mathbb{P}/d\mathbb{P} = \tilde{H}_T$, Girsanov theorem states that

$$\tilde{W}_t = \hat{W}_t + \int_0^t \frac{\dot{\mu}_v}{\sigma_S} dv, \quad 0 \leq t \leq T.$$
is a Brownian Motion under \((\bar{P}, (\mathcal{F}^S_t)_{0 \leq t \leq T})\).

Then we can rewrite the wealth process dynamics as:

\[
X_T + \int_0^T c_v dv = x + \int_0^T \pi_v \sigma_S d\tilde{W}_v,
\]

Since we have \(X_T \geq 0\), it’s easy to see that \(\int_0^T \pi_v \sigma_S d\tilde{W}_v\) is a supermartingale under \((\Omega, \bar{P}, \mathbb{F}^S)\), and take the expectation under \(\bar{P}\), we have:

\[
x \geq \mathbb{E} \left[ \int_0^T c_v dv \right].
\]

Follow the inequality (5.2), we will further have:

\[
x \geq z \mathbb{E} \left[ \int_0^T \exp \left( \int_0^t (\delta(u) - \alpha(u)) du \right) dv \right].
\]

Since \(\delta(t)\) and \(\alpha(t)\) are deterministic functions, we easily arrive \(x \geq w(0)z\).

In general, for \(\forall t \in [0, T]\), follow the same procedure, we can then take conditional expectation under filtration \(\mathcal{F}^S_t\), and get

\[
X_t > Z_t \mathbb{E} \left[ \int_t^T \exp \left( \int_t^\tau (\delta(u) - \alpha(u)) du \right) dv \mid \mathcal{F}^S_t \right],
\]

again since \(\delta(t), \alpha(t)\) are deterministic, we obtain \(X_t \geq w(t)Z_t, \quad 0 \leq t \leq T\).

\[\square\]

5.1. The Case \(p < 0\).

(The PROOF OF THEOREM 4.2).

First, for any pair of admissible control \((\pi_t, c_t) \in \mathcal{A}\), denote \(G^{\pi_t,c_t}\) as the generator of the process \(\tilde{V}(t, x, z, \mu, \nu)\) under control \((\pi_t, c_t)\). Ito’s lemma gives

\[
d\tilde{V}(t, x, z, \mu, \nu) = \left[G^{\pi_t,c_t}\tilde{V}(t, x, z, \mu, \nu)\right] dt + \left[\tilde{V}_x \sigma_S \pi_t + \tilde{V}_z + \tilde{V}_\mu \left(\hat{\Omega}(t) + \sigma_S \pi_t \rho \right) / \sigma_S\right] d\tilde{W}_t,
\]

Recall \(\tilde{V}(t, x, z, \eta)\) is the classical solution of HJB equation (4.13), choose the localizing sequence \(\tau_n\), we integrate the equation (5.3) on \([t, \tau_n \wedge T]\), and take the expectation under probability measure \(\mathbb{P}_{x,z,\eta}\), and let’s denote \(E = E_{x,z,\eta}\), we have

\[
\tilde{V}(t, x, z, \eta) \geq \mathbb{E} \left[ \int_t^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds \right] + \mathbb{E} \left[ \tilde{V}(\tau_n \wedge T, \pi_{\tau_n \wedge T}, \nu_{\tau_n \wedge T}, \mu_{\tau_n \wedge T}) \right].
\]

Now, we follow the idea by Janeček and Sîrbu [9], let’s fix this pair of control choice \((\pi_t, c_t) \in \mathcal{A} = \mathcal{A}_x\), where we denote \(\mathcal{A}_x\) as the admissible space with initial endowment \(x\). And for \(\forall \epsilon > 0\), it is clear that \(\mathcal{A}_x \subseteq \mathcal{A}_{x+\epsilon}\), and \((\pi_t, c_t) \in \mathcal{A}_{x+\epsilon}\). Also it is clear that \(X_t^{x+\epsilon} = X_t^x + \epsilon = X_t + \epsilon, \quad 0 \leq t \leq T\). Follow the same procedure above, and notice process \(Z_t\) keeps the same under the consumption policy \(c_t\), then under probability measure \(\mathbb{P}_{x,z,\eta}\), we can obtain:

\[
\tilde{V}(t, x + \epsilon, z, \eta) \geq \mathbb{E} \left[ \int_t^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds \right] + \mathbb{E} \left[ \tilde{V}(\tau_n \wedge T, \pi_{\tau_n \wedge T} + \epsilon, Z_{\tau_n \wedge T}, \nu_{\tau_n \wedge T}) \right].
\]
By Monotone Convergence Theorem, we first know:

\[
\lim_{n \to +\infty} \mathbb{E} \left[ \int_0^{\tau_n \wedge T} \left( \frac{c_s - Z_s}{p} \right)^p ds \right] = \mathbb{E} \left[ \int_0^T \left( \frac{c_s - Z_s}{p} \right)^p ds \right].
\]

For simplicity, let’s denote \( Y_t = \left( X_t - w(t)Z_t \right) \), we know by definition \([4.36]\) that:

\[
\tilde{V}(\tau_n \wedge T, X_{\tau_n \wedge T} + \epsilon, \epsilon_{\tau_n \wedge T}, \bar{\mu}_{\tau_n \wedge T}) = \frac{1}{p} (Y_{\tau_n \wedge T} + \epsilon)^p N_{\tau_n \wedge T}. \]

Proposition \([5.1]\) gives \( X_t \geq w(t)Z_t \) for \( 0 \leq t \leq T \) under any admissible control pair \((\pi_t, c_t)\), we know \( Y_{\tau_n \wedge T} + \epsilon \geq \epsilon > 0, \) \( \forall 0 \leq t \leq T \). Since also \( p < 0 \), we will have

\[
\sup_n (Y_{\tau_n \wedge T} + \epsilon)^p < e^p < +\infty.
\]

Now from Remark \([A.1]\) we already derived that \( A(t; s) \leq 0, \) \( \forall 0 \leq t \leq s \leq T \). Combining this with the fact that \( w(s), \delta(s) \) are continuous and hence bounded on \([0, T]\) and when \( p < 0 \), we also have \( 1 - a(t)\tilde{\Omega}(t) > 0 \) and \( 1 - f(t)\tilde{\Omega}(t) > 0 \) as well as \( a(t; s), b(t; s), c(t; s), f(t; s) \) and \( g(t; s) \) are all bounded for \( 0 \leq t \leq s \leq T \), we deduce that the explicit solutions \( B(t; s) \) and \( C(t; s) \) are both bounded on \([0, T]\), hence we have:

\[
N(t, \eta) \leq k_1 \exp(k\eta) \text{ for some large constants } k, k_1 > 1,
\]

which shows the existence of some constants \( \bar{k}, \bar{k}_1 > 1 \) such that

\[
\sup_n N_{\tau_n \wedge T}^{1-p} \leq \sup_{t \in [0, T]} \left( k_1 \exp \left( k\bar{\mu}_t \right) \right)^{1-p} \leq \bar{k}_1 \exp \left( \bar{k} \sup_{t \in [0, T]} \bar{\mu}_t \right).
\]

We recall that \( \bar{\mu}_t \) satisfies the Ornstein Uhlenbeck diffusion \([4.2]\), which gives:

\[
\bar{\mu}_t = e^{-\lambda t} \eta + \bar{\mu}(1 - e^{-\lambda t}) + \int_0^t e^{\lambda(u-t)} \frac{\tilde{\Omega}(u) + \sigma_S \sigma_{\mu \rho}}{\sigma_S} d\tilde{W}_u.
\]

Hence, there exists positive constants \( l \) and \( l_1 > 1 \) large enough, such that:

\[
\sup_{t \in [0, T]} \bar{\mu}_t \leq l + \sup_{t \in [0, T]} l_1 \tilde{W}_t, \quad t \in [0, T].
\]

Using the exponential distribution of maximum of the Brownian Motion, there exists some positive constants \( \bar{l} > 1 \) and \( \bar{l}_1 \) such that

\[
\mathbb{E} \left[ \sup_n N_{\tau_n \wedge T}^{1-p} \right] \leq \bar{l}_1 \mathbb{E} \left[ \exp \left( \sup_{t \in [0, T]} l_1 \tilde{W}_t \right) \right] < +\infty.
\]

At last, by the above \([5.7]\) and \([5.8]\), we can conclude that

\[
\mathbb{E} \left[ \sup_n \tilde{V}(\tau_n \wedge T, X_{\tau_n \wedge T}, Z_{\tau_n \wedge T}, \bar{\mu}_{\tau_n \wedge T}) \right] < +\infty.
\]

By virtue of Dominated Convergence Theorem, we can deduce:

\[
\lim_{n \to +\infty} \mathbb{E} \left[ \tilde{V}(\tau_n \wedge T, X_{\tau_n \wedge T} + \epsilon, Z_{\tau_n \wedge T}, \bar{\mu}_{\tau_n \wedge T}) \right] = \mathbb{E} \left[ \frac{1}{p} (Y_T + \epsilon)^p N(T, \bar{\mu}_T) \right] = \mathbb{E} \left[ \frac{(X_T + \epsilon)^p}{p} \right] > \mathbb{E} \left[ \frac{X_T^p}{p} \right].
\]
Combine this with equation (5.5), and notice the pair of control \((\pi_t, c_t) \in \mathcal{A}\), we will see that:

\[
\tilde{V}(t, x + \epsilon, z, \eta) \geq \sup_{\pi, c \in \mathcal{A}} \mathbb{E}\left[\int_t^T (c_s - Z_s)^p \frac{d\tilde{W}_t}{p} + \frac{X^p_t}{p}\right] = V(t, x, z, \eta).
\]

Notice \(\tilde{V}(t, x + \epsilon, z, \eta)\) is continuous in variable \(x\), and since \(\epsilon > 0\) is arbitrary, we can take the limit as:

\[
\tilde{V}(t, x, z, \eta) = \lim_{\epsilon \to 0} \tilde{V}(t, x + \epsilon, z, \eta) \geq V(t, x, z, \eta).
\]

On the other hand, for \(\pi_t^*\) and \(c_t^*\) defined by (4.42) and (4.43) respectively, we first want to show the SDE for wealth process:

\[
(5.9) \quad dX^*_t = (\pi_t^* \mu_t - c_t^*) dt + \sigma S \pi_t^* d\tilde{W}_t, \quad 0 \leq t \leq T,
\]

with initial condition \(x > w(0)\), \(\pi\) has an unique strong solution and also satisfies \(X^*_t > w(t)Z^*_t, \forall s \in [0, T]\).

Denote \(Y_t^* = X_t^* - w(t)Z_t^*\), and apply Ito’s lemma and substitute \(\pi_t^*\) as defined by (4.42), we can get:

\[
(5.10) \quad dY^*_t = \left[\pi_t^* \mu_t - c_t^* - w_t(t)Z^*_t - w_t(t)\delta(t)c_t^* + w(t)\alpha(t)Z^*_t\right] dt + \pi_t^* \sigma S d\tilde{W}_t
\]

\[
= \left[-w_t(t) + w(t)\alpha(t)\right] Z^*_t - (1 + w(t)\delta(t))c_t^* + \frac{\mu^2}{(1 - p)\sigma^2} Y^*_t
\]

\[
+ \frac{\hat{\Omega}(t) + \sigma S \mu \rho}{\sigma^2} \frac{N_t \hat{\mu} Y_t^*}{N} dt + \frac{\hat{\Omega}(t) + \sigma S \mu \rho}{\sigma^2} \frac{N_t \hat{\mu} Y_t^*}{N} \tilde{W}_t.
\]

Recall the definition of \(w(t)\) by (4.18) and substitute \(c_t^*\) defined by (4.43) into (5.10) above, we will further have

\[
dY^*_t = \left[-\left(1 + \delta(t)w(t)\right)^{\frac{p}{1-p}} + \frac{\hat{\mu}^2}{(1 - p)\sigma^2} + \frac{\hat{\Omega}(t) + \sigma S \mu \rho}{\sigma^2} \frac{N_t \hat{\mu} Y_t^*}{N}\right] Y_t^* dt
\]

\[
+ \left[\frac{\hat{\mu}}{(1 - p)\sigma} + \frac{\hat{\Omega}(t) + \sigma S \mu \rho}{\sigma} \frac{N_t \hat{\mu} Y_t^*}{N}\right] Y_t^* \tilde{W}_t.
\]

In order to solve \(X^*_t\) in a more explicit formula, we define the auxiliary process by:

\[
\Gamma_t = \frac{N_t(\hat{\mu} - \tilde{\mu})}{Y_t^*}, \quad \forall 0 \leq t \leq T.
\]

By Ito’s lemma, we can derive the SDE for process \(\Gamma_t\) as:

\[
(5.11) \quad d\Gamma_t = \frac{\Gamma_t}{N} \left[N_t - \lambda(\hat{\mu} - \tilde{\mu})N_t + \frac{\hat{\Omega}(t) + \sigma S \mu \rho}{2\sigma^2} \frac{N_t}{N} \frac{\hat{\mu}^2}{(1 - p)\sigma^2} + \frac{\hat{\Omega}(t) + \sigma S \mu \rho}{\sigma^2} \frac{N_t}{N} \right] dt + \frac{\hat{\mu}}{N} \left[\frac{-\hat{\mu}}{(1 - p)\sigma}\right] d\tilde{W}_t.
\]
Recall that \( N(t, \eta) \) satisfies the linear PDE (4.23), we can simplify (5.11) to be:

\[
d\Gamma_t = \Gamma_t \left[ \frac{\hat{p}^2}{2(1-p)^2\sigma_S^2} \right] dt + \Gamma_t \left[ -\frac{\hat{\mu}_t}{(1-p)\sigma_S} \right] d\hat{W}_t.
\]

Hence, we can finally get the above SDE has an unique strong solution as:

\[
\Gamma_t = \Gamma_0 \exp \left( -\int_0^t \frac{\hat{\mu}_u^2}{2(1-p)^2\sigma_S^2} du - \int_0^t \frac{\hat{\mu}_u}{(1-p)\sigma_S} d\hat{W}_u \right).
\]

Initial condition \( \Gamma_0 = \frac{N(0, \eta)}{x-w(0)\tau} > 0 \) implies \( \Gamma_t > 0, \forall 0 \leq t \leq T \). And, hence, we finally proved that the SDE (5.9) has an unique strong solution defined by (4.44) and the solution \( X^*_t \) satisfies the wealth process constraint (5.1).

Now, we proceed to verify \( \pi^*_t \) and \( c^*_t \) are actually in the admissible space \( \mathcal{A} \).

First, by the definition (4.42) and (4.43), it’s clear that \( \pi^*_t \) and \( c^*_t \) are \( \mathcal{F}^S_t \) progressively measurable, and by the path continuity of \( Y^*_t = X^*_t - w(t)Z^*_t \), hence, of \( \pi^*_t \) and \( c^*_t \), it’s easy to show that:

\[
\int_0^T (\pi^*_t)^2 dt < +\infty, \quad \text{and} \quad \int_0^T c^*_t dt < +\infty, \quad \text{a.s.}
\]

Also, since \( X^*_t > w(t)Z^*_t, \forall s \in [0, T] \), by the definition of \( c^*_t \), we know the consumption constraint \( c^*_t > Z^*_t, \forall 0 \leq t \leq T \) is satisfied. And hence \( (\pi^*_t, c^*_t) \in \mathcal{A} \).

Given the pair of control policy \( (\pi^*_t, c^*_t) \) as above, following the same steps and the definition of stopping time \( \tau_n \), instead of (5.4), we can now instead get the equality:

\[
\tilde{V}(t, x, z, \eta) = \mathbb{E} \left[ \int_{t}^{\tau_n \wedge T} \left( \frac{c^*_t - Z^*_t}{p} \right)^p dt \right] + \mathbb{E} \left[ \tilde{V}(\tau_n \wedge T, X^*_{\tau_n \wedge T}, Z^*_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T}) \right].
\]

And hence, we apply Monotone Convergence Theorem again:

\[
\lim_{n \to +\infty} \mathbb{E} \left[ \int_{t}^{\tau_n \wedge T} \left( \frac{c^*_t - Z^*_t}{p} \right)^p dt \right] = \mathbb{E} \left[ \int_{t}^{T} \left( \frac{c^*_t - Z^*_t}{p} \right)^p dt \right],
\]

When \( p < 0 \), we have function \( \tilde{V}(t, x, z, \eta) < 0 \) by it’s definition, and by Fatou’s lemma,

\[
\lim_{n \to +\infty} \mathbb{E} \left[ \tilde{V}(\tau_n \wedge T, X^*_{\tau_n \wedge T}, Z^*_{\tau_n \wedge T}; \hat{\mu}_{\tau_n \wedge T}) \right] \leq \mathbb{E} \left[ \tilde{V}(T, X^*_T, Z^*_T; \hat{\mu}_T) \right] = \mathbb{E} \left[ \left( \frac{X^*_T}{p} \right)^p \right] \cdot \mathbb{E} \left[ \left( \frac{X^*_T}{p} \right)^p \right].
\]

Therefore, it gives

\[
\tilde{V}(t, x, z, \eta) = \mathbb{E} \left[ \int_{t}^{T} \left( \frac{c^*_t - Z^*_t}{p} \right)^p dt + \frac{X^*_p}{p} \right] \leq V(t, x, z, \eta)
\]

which completes the whole proof.

\[
\square
\]

5.2. The Case: \( 0 < p < 1 \).

We proceed to prove the following two Lemmas which play important roles in the proof of the second part of our main result.

**Lemma 5.1.** If constant \( k > 0 \) satisfies:

\[
k < \frac{\lambda^2 \sigma_S^2}{2(\Theta + \sigma_S \sigma_\mu \rho)^2}
\]
there exists a constant $\Lambda_1$ independent of $t$, such that
\[
E\left[\exp\left(\int_0^t k\mu_s^2 ds\right)\right] \leq \Lambda_1 < +\infty, \quad \forall t \in [0, T].
\]

Proof. It is easy to choose an increasing sequence of smooth functions $Q_n(y) \nearrow ky^2$ as $n \to \infty$ such that $0 \leq Q_n(y) \leq n$ with $|Q'_n(y)|$ and $|Q''_n(y)|$ uniformly bounded. And for each fixed $t \in [0, T]$ and $\eta$, we define:
\[
\phi(t, \eta) = E\left[\exp\left(\int_0^t Q_n(\mu_s) ds\right)\right],
\]
where $\mu_0 = \eta$.

Similar to the proof of Feynman-Kac formula, the function $\phi(t, \eta)$ is a classical solution of the linear parabolic equation:
\[
(5.13) \quad \phi_t = \left(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\right)^2 \frac{\phi_{yy}}{2\sigma_S^2} - \lambda(\eta - \bar{\mu})\phi_y + Q_n(y)\phi,
\]
with initial condition $\phi(0, \eta) = 1$. See also Lemma 1.12 in Pang [14] for details.

First, it’s clear that constant 0 is a subsolution of the above equation. Moreover, under assumption (5.12), it’s easy to show that for each $t \in [0, T]$, the equation:
\[
\frac{2(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho)^2}{\sigma_S^2} x^2 - 2\lambda x + k = 0
\]
has two positive real roots $x_1 = \frac{-\sqrt{\lambda^2 - \frac{2(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}} - k}{2(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho)^2}$ and $x_2 = \frac{\sqrt{\lambda^2 - \frac{2(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}} + k}{2(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho)^2}$. And for any positive constant $a$ such that:
\[
0 < a < \frac{\lambda + \sqrt{\lambda^2 - \frac{(\Theta_1 + \sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}}}{2(\Theta_1 + \sigma_S\sigma_\mu\rho)^2}
\]
with $\Theta_1 = \max(\theta, \theta^*)$ and $\Theta_2 = \min(\theta, \theta^*)$, and the positive constant $b$ such that:
\[
b > a\left(\frac{\Theta_1 + \sigma_S\sigma_\mu\rho}{\sigma_S^2}\right)^2 - \frac{\lambda^2 \mu^2 a^2}{2a^2(\Theta_1 + \sigma_S\sigma_\mu\rho)^2} - 2a\lambda + k,
\]

it’s easy to verify that $f(t, \eta) = \exp(bt + a\eta^2)$ satisfies:
\[
f_t \geq \left(\frac{\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho}{2\sigma_S^2}\right)^2 f_{yy} - \lambda(\eta - \bar{\mu})f_y + k\eta^2,
\]
with the initial condition $f(0, \eta) \geq 1$.

And since $Q_n(\eta) < k\eta^2$, we get function $f(t, \eta)$ is the supersolution of the equation (5.13), and $\langle 0, f(t, \eta) \rangle$ is the coupled subsolution and supersolution. Theorem 7.2 from Pao [15] shows that function $\phi(t, \eta)$ satisfies: $0 \leq \phi(t, \eta) \leq f(t, \eta) \leq e^{bT + a\eta^2} \equiv \Lambda_1$, and hence Monotone Convergence Theorem leads to:
\[
E\left[\exp\left(\int_0^t k\mu_s^2 ds\right)\right] \leq \Lambda_1 < +\infty, \quad \forall t \in [0, T].
\]
Lemma 5.2. If constant $\bar{k} > 0$ satisfies
\begin{equation}
\bar{k} < \frac{\lambda \sigma_S^2}{(\Theta + \sigma_S \sigma_\mu \rho)^2},
\end{equation}
for fixed constant $\kappa > 0$, there exists a constant $\Lambda_2$ independent of $t$, and
\[E\left[\exp\left(\bar{k}(\hat{\mu}_t + \kappa)^2\right)\right] \leq \Lambda_2 < \infty, \quad t \in [0, T].\]

Proof. Similar to the proof of Lemma 5.1, we again construct an increasing sequence of functions $\{Q_n(y)\}$ for $n \in \mathbb{N}$ such that $\lim_{n \to +\infty} Q_n(y) = \bar{k}(y + \kappa)^2$. And for each fixed $t \in [0, T]$ and $\eta$, we define:
\[\psi(t, \eta) = E\left[\exp\left(Q_n(\hat{\mu}_t)\right)\right],\]
where $\hat{\mu}_0 = \eta$.

Then a direct corollary of Theorem 5.6.1 of Friedman [8] gives the function $\psi(t, \eta)$ is a classical solution of the linear parabolic equation:
\begin{equation}
\psi_t = \frac{\left(\Omega(t) + \sigma_S \sigma_\mu \rho\right)^2}{2\sigma_S^2} \psi_{\eta\eta} - \lambda(\eta - \bar{\mu})\psi_\eta,
\end{equation}
with initial condition $\psi(0, \eta) = e^{Q_n(\eta)}$.

Under assumption (5.14), and choose any constant $a$ such that
\[\bar{k} < a < \frac{\lambda \sigma_S^2}{(\Theta + \sigma_S \sigma_\mu \rho)^2},\]
where $x_1 = \frac{\lambda \sigma_S^2}{(\Omega(t) + \sigma_S \sigma_\mu \rho)^2}$ is one real root of the algebraic equation:
\[2\left(\frac{\Omega(t) + \sigma_S \sigma_\mu \rho}{\sigma_S^2}\right)^2 x^2 - 2\lambda x = 0\]
for each fixed $t \in [0, T]$. And choose any positive constant $b$ such that
\[b > a \left(\frac{\Theta + \sigma_S \sigma_\mu \rho}{\sigma_S^2}\right)^2 + \frac{2a^2(\Theta + \sigma_S \sigma_\mu \rho)^2 \kappa^2}{\sigma_S^2} + 2a\lambda \bar{\mu} \kappa - \frac{2a^2(\Theta + \sigma_S \sigma_\mu \rho)^2 - a\lambda \kappa - a\lambda \bar{\mu}}{2a^2(\Theta + \sigma_S \sigma_\mu \rho)^2},\]

It is easy to verify that $f(t, \eta) = \exp(bt + a(\eta + \kappa)^2)$ satisfies
\[f_t \geq \frac{\left(\Omega(t) + \sigma_S \sigma_\mu \rho\right)^2}{2\sigma_S^2} f_{\eta\eta} - \lambda(\eta - \bar{\mu})f_\eta,\]
with the initial condition $f(0, \eta) = e^{a(\eta + \kappa)^2} \geq \psi(0, \eta)$, hence we get the function $f(t, \eta)$ is the supersolution of the equation (5.15), and it is trivial to show $g(t, \eta) \equiv 0$ is the subsolution, therefore $\langle 0, f(t, \eta) \rangle$ are the coupled subsolution and supersolution. Again by Theorem 7.2 from Pao [15], that function $\psi(t, \eta)$ satisfies: $0 \leq \psi(t, \eta) \leq f(t, \eta) \leq e^{bT + a(\eta + \kappa)^2} \equiv \Lambda_2$, hence Monotone
Convergence Theorem implies:

\[ E \left[ \exp \left( \hat{k}(\hat{\mu} + \kappa)^2 \right) \right] \leq \Lambda_2 < +\infty, \quad \forall t \in [0, T]. \]

\[ \square \]

(The PROOF OF THEOREM 4.2 CONTINUED).

For any pair of admissible control \((\pi_t, c_t) \in A\), similar to the case for \(p < 0\), choose the same localizing sequence \(\tau_n\) such that

\[ \tilde{V}(t, x, z, \eta) \geq E \left[ \int_t^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds \right] + E \left[ \tilde{V}(\tau_n \wedge T, X_{\tau_n \wedge T}, Z_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T}) \right]. \]

Now, by monotone convergence theorem, we first know:

\[ \lim_{n \to +\infty} \left[ \frac{\tau_n \wedge T}{E} \int_t^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds \right] = E \left[ \int_t^T \frac{(c_s - Z_s)^p}{p} ds \right]. \]

And for \(0 < p < 1\), \(\tilde{V}(t, X_t, Z_t, \hat{\mu}_t) \geq 0\) for all \(t \in [0, T]\) by the definition (4.36) and (5.1), and Fatou’s lemma yields that:

\[ \lim_{n \to +\infty} E \left[ \tilde{V}(\tau_n \wedge T, X_{\tau_n \wedge T}, Z_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T}) \right] \geq E \left[ \tilde{V}(T, X_T, Z_T, \hat{\mu}_T) \right] = E \left[ \frac{X_T}{p} \right], \]

which implies that:

\[ \tilde{V}(t, x, z, \eta) \geq \sup_{\pi, c \in A} E \left[ \int_t^T \frac{(c_s - Z_s)^p}{p} ds + \frac{X_T}{p} \right] = V(t, x, z, \eta). \]

On the other hand, for the \(\pi_t^*\) and \(c_t^*\) defined by (4.42), (4.43), again follow the same procedure in the proof for case \(p < 0\), we can show \(\pi_t^*\) and \(c_t^*\) are actually in the admissible space \(A\).

Now, by policies \(\pi_t^*\) and \(c_t^*\), similarly, we can now get the equality:

\[ \tilde{V}(t, x, z, \eta) = E \left[ \int_t^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds \right] + E \left[ \tilde{V}(\tau_n \wedge T, X_{\tau_n \wedge T}, Z_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T}) \right]. \]

By the definition of \(\tilde{V}(t, x, z, \eta)\) and Holder inequality, we know that:

\[ E \left[ \tilde{V}(T \wedge \tau_n, X_{T \wedge \tau_n}, Z_{T \wedge \tau_n}, \hat{\mu}_{T \wedge \tau_n}) \right] \leq k_1 E \left[ \left( \frac{Y^{\pi}_{T \wedge \tau_n}}{N} \right)^p N(T \wedge \tau_n, \hat{\mu}_{T \wedge \tau_n}) \right] \]

\[ \leq k_2 \left( E \left[ \exp \left( \int_0^{T \wedge \tau_n} \frac{2p\hat{\mu}_u}{(1-p)\sigma S} d\tilde{W}_u - \int_0^{T \wedge \tau_n} \frac{2p^2\hat{\mu}_u}{(1-p)^2\sigma^2 S} du \right) \right] \right)^{\frac{1}{2}} \]

\[ \cdot \left( E \left[ \exp \left( \int_0^{T \wedge \tau_n} \frac{(p^2 + p)\hat{\mu}_u^2}{(1-p)^2\sigma^2 S} du \right) N^2(T \wedge \tau_n, \hat{\mu}_{T \wedge \tau_n}) \right] \right)^{\frac{1}{2}} \]

for some positive constants \(K_1, K_2\), which are independent of \(n\).

Notice that \(\exp \left( \int_0^{T \wedge \tau_n} \frac{2p\hat{\mu}_u}{(1-p)\sigma S} d\tilde{W}_u - \int_0^{T \wedge \tau_n} \frac{2p^2\hat{\mu}_u}{(1-p)^2\sigma^2 S} du \right)\) is a positive local martingale, therefore, a supermartingale, we can apply the bounded stopping theorem and get:

\[ E \left[ \exp \left( \int_0^{T \wedge \tau_n} \frac{2p\hat{\mu}_u}{(1-p)\sigma S} d\tilde{W}_u - \int_0^{T \wedge \tau_n} \frac{2p^2\hat{\mu}_u}{(1-p)^2\sigma^2 S} du \right) \right] \leq 1. \]
Thus, we can get:
\[
V(t, x, z, \eta) \leq \mathbb{E}\left[ \int_t^{\tau_n \wedge T} \frac{(c_d - z_s)^p}{p} ds \right] + \left( \mathbb{E}\left[ \exp\left( \int_0^{T \wedge \tau_n} \frac{(p^2 + p)\mu_s^2}{(1 - p)^2 \sigma_S^2} du \right) \right] \right)^{\frac{1}{2}}.
\]
And now, we observe that:
\[
\sup_n \left[ \mathbb{E}\left[ \exp\left( \int_0^{T \wedge \tau_n} \frac{(p^2 + p)\mu_s^2}{(1 - p)^2 \sigma_S^2} du \right) \right] \sup_n \left[ \mathbb{E}\left[ N^2(T \wedge \tau_n, \hat{\mu}_T \wedge \tau_n) \right] \right] \right] \leq \mathbb{E}\left[ \exp\left( \int_0^{T \wedge \tau_n} \frac{(p^2 + p)\mu_s^2}{(1 - p)^2 \sigma_S^2} du \right) \right] + \mathbb{E}\left[ \sup_n \left[ N^4(T \wedge \tau_n, \hat{\mu}_T \wedge \tau_n) \right] \right].
\]
Hence, we have:
\[
\mathbb{E}\left[ \sup_n \left[ \exp\left( \int_0^{T \wedge \tau_n} \frac{(p^2 + p)\mu_s^2}{(1 - p)^2 \sigma_S^2} du \right) \right] \sup_n \left[ \mathbb{E}\left[ N^2(T \wedge \tau_n, \hat{\mu}_T \wedge \tau_n) \right] \right] \right] \leq \mathbb{E}\left[ \exp\left( \int_0^{T \wedge \tau_n} \frac{(p^2 + p)\mu_s^2}{(1 - p)^2 \sigma_S^2} du \right) \right] + \mathbb{E}\left[ \sup_n \left[ N^4(T \wedge \tau_n, \hat{\mu}_T \wedge \tau_n) \right] \right].
\]

We first recall that under Assumption (4.33), Lemma 4.1 implies that there exists constants $k$, $k_1$ such that
\[
N(t, \eta) \leq ke^{K_1(\eta + k_1)^2},
\]
where $A(t; s) \leq \bar{K}_1$ for all $0 \leq t \leq s \leq T$, and hence we have
\[
\sup_n \left( \mathbb{E}\left[ \sup_{t \in [0, T]} \left[ N^4(T \wedge \tau_n, \hat{\mu}_T \wedge \tau_n) \right] \right] \right) \leq \sup_{t \in [0, T]} ke^{4\bar{K}_1(\hat{\mu}_t + k_1)^2}.
\]

Then we just need to show that
\[
(5.17) \quad \mathbb{E}\left[ \sup_{t \in [0, T]} ke^{4\bar{K}_1(\hat{\mu}_t + k_1)^2} \right] < \infty.
\]
Define $\varphi(x) \triangleq e^{4\bar{K}_1(x + k_1)^2}$ and apply Ito’s lemma, we have
\[
d\varphi(\hat{\mu}_t) = \varphi(\hat{\mu}_t) \left[ -8K_1 \lambda - 8K_2 \left( \frac{\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho}{\sigma_S^2} \right)^2 \right] \mu_t^2 + 8K_1 \lambda \hat{\mu}_t \mu_t - 4K_1 \left( \frac{\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho}{\sigma_S^2} \right)^2 dt + dL_t
\]
\[
\leq \varphi(\hat{\mu}_t)k_2 dt + dL_t,
\]
where $k_2 > 0$ is a upper bound constant, and the local martingale part is:
\[
dL_t \triangleq \varphi(\hat{\mu}_t)8K_1 \hat{\mu}_t \left( \frac{\hat{\Omega}(t) + \sigma_S \sigma_{\mu} \rho}{\sigma_S} \right) d\hat{W}_t.
\]

From which we can derive that
\[
\mathbb{E}\left[ \sup_{t \in [0, T]} \varphi(\hat{\mu}_t) \right] \leq \varphi(\eta) + \int_0^T k_2 \mathbb{E}\left[ \sup_{s \in [0, t]} \varphi(\hat{\mu}_s) \right] dt + \mathbb{E}\left[ \sup_{t \in [0, T]} L_t \right],
\]
Burkholder-Davis-Gundy Inequality and Jensen’s Inequality imply that
\[
\mathbb{E}\left[\sup_{t \in [0,T]} L_t \right] \leq k_3 \left( \mathbb{E}\left[ \int_0^T 64 \tilde{K}_1^2 \left( \frac{\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho}{\sigma_S^2} \right)^2 \tilde{\mu}_t^2 e^{8\tilde{K}_1(\hat{\mu}_t+k_1)^2} dt \right] \right)^{\frac{1}{2}},
\]

However, under Assumption (4.40) and by Lemma 5.2, there exists a constant \( \Lambda_2 \) such that
\[
\mathbb{E}\left[ \hat{\mu}_t^2 e^{8\tilde{K}_1(\hat{\mu}_t+k_1)} \right] \leq \mathbb{E}\left[ e^{(8\tilde{K}_1+1)(\hat{\mu}_t+k_1)^2} \right] \leq \Lambda_2 < \infty, \quad t \in [0,T].
\]

Hence we obtain the boundedness \( \mathbb{E}\left[ \sup_{t \in [0,T]} L_t \right] \leq k_4 < \infty \) for some constant \( k_4 \), and
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \varphi(\hat{\mu}_t) \right] \leq \varphi(\eta) + \int_0^T k_2 \mathbb{E}\left[ \sup_{s \in [0,t]} \varphi(\hat{\mu}_s) \right] dt + k_4,
\]

The Gronwall’s Inequality verifies (5.17).

For the first part, by coefficients Assumption (4.39), we can therefore apply Lemma 5.1 and it yields that:
\[
\mathbb{E}\left[ \exp \left( \int_0^T \frac{(2p^2 + 2p)\hat{\mu}_u^2}{(1-p)^2\sigma_S^2} du \right) \right] < \Lambda_1 < +\infty,
\]
for some constant \( \Lambda_1 > 0 \). We can then get that:
\[
\sup_n \mathbb{E}\left[ \widetilde{V}(\tau_n \wedge T, X_{\tau_n \wedge T}, Z_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T}) \right] < \Lambda < \infty
\]
for some constant \( \Lambda > 0 \), independent of \( n \). Therefore, Dominated Convergence Theorem leads to:
\[
\lim_{n \to +\infty} \mathbb{E}\left[ \widetilde{V}(\tau_n \wedge T, X_{\tau_n \wedge T}, Z_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T}) \right] = \mathbb{E}\left[ \frac{(X_T^*)^p}{p} \right],
\]

which completes the whole proof. \( \Box \)

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Appendix A. Fully Explicit Solutions

Follow the arguments by Kim and Omberg [10], we can even solve the auxiliary ODEs (4.27), (4.28), (4.29), (4.30) and (4.31) fully explicitly depending on the risk aversion constant \( p \) and all the market coefficients \( \sigma_S, \sigma_\mu, \lambda, \rho \).
A.1. The Normal Solution. The condition for the Normal solution is

\[(A.1) \quad \Delta \triangleq \lambda^2 - \frac{2\lambda \rho \sigma_{\mu}}{(1 - p)\sigma_{S}^2} - \frac{\rho \sigma_{\mu}^2}{(1 - p)\sigma_{S}^2} > 0,\]

and then we define:

\[
\xi = \sqrt{\Delta} = \sqrt{\gamma_2^2 - \gamma_1 \gamma_3}, \quad \gamma_1 = \frac{(1 - p + \rho \rho^2)}{1 - p} \sigma_{\mu}^2, \\
\gamma_2 = -\lambda + \frac{p \rho \sigma_{\mu}}{(1 - p)\sigma_{S}}, \quad \gamma_3 = \frac{p}{(1 - p)\sigma_{S}^2}, \\
\xi_1 = \frac{\sqrt{(1 - \rho^2)\sigma_{S}^2 + (\lambda \sigma_{S} + \rho \sigma_{\mu})^2}}{\sigma_{S}}.
\]

We can solve the equations \((4.27), (4.28), (4.29), (4.30)\) and \((4.31)\) as:

\[
a(t; s) = \frac{p (1 - e^{2\xi (t-s)})}{2(1 - p)\sigma_{S}^2 [2\xi - (\xi + \gamma_2)(1 - e^{2\xi (t-s)})]},
\]

\[
b(t; s) = \frac{p \lambda \mu (1 - e^{\xi (t-s)})^2}{2(1 - p)\sigma_{S}^2 \xi [2\xi - (\xi + \gamma_2)(1 - e^{2\xi (t-s)})]},
\]

\[
c(t; s) = \frac{p}{2(1 - p)\sigma_{S}^2} \left( \frac{\lambda^2 \mu^2}{\xi^2} - \frac{\sigma_{\mu}^2}{\xi + \gamma_2} \right) (s - t) + \frac{p \lambda^2 \mu^2}{2(1 - p)\sigma_{S}^2 \xi^3} \left[ \gamma_1 \right] \frac{2\xi - (\xi + \gamma_2)(1 - e^{2\xi (t-s)})}{2\xi}.
\]

\[
f(t; s) = -\frac{1}{2\sigma_S} \left( \frac{(\sigma_S \xi_1 + \lambda \sigma_{S} + \rho \sigma_{\mu}) + (\sigma_S \xi_1 - \lambda \sigma_{S} - \rho \sigma_{\mu}) e^{2\xi_1 (t-s)}}{\sigma_S \xi_1 e^{\xi_1 (t-s)}} \right) = \frac{(1 - p)(1 - \rho^2)}{2(1 - p + pp^2)} \log \left( \frac{\sigma_S \xi + \lambda \sigma_{S} + \rho \sigma_{\mu}}{\sigma_S \xi - \lambda \sigma_{S} - \rho \sigma_{\mu}} \right) + \sigma_S \xi e^{\xi_1 (t-s)} - \sigma_S \xi e^{\xi (t-s)}.
\]

\[
g(t; s) = \frac{1}{2} \log \left( \frac{(\sigma_S \xi_1 + \lambda \sigma_{S} + \rho \sigma_{\mu}) + (\sigma_S \xi_1 - \lambda \sigma_{S} - \rho \sigma_{\mu}) e^{2\xi_1 (t-s)}}{2\sigma_S \xi_1 e^{\xi_1 (t-s)}} \right) = \frac{1}{2(1 - p + pp^2)} \log \left( \frac{(\sigma_S \xi + \lambda \sigma_{S} + \rho \sigma_{\mu}}{\sigma_S \xi - \lambda \sigma_{S} - \rho \sigma_{\mu}} \right) + \sigma_S \xi e^{\xi_1 (t-s)} - \sigma_S \xi e^{\xi (t-s)}.
\]

The condition for the bounded Normal solution is

\[(A.2) \quad \gamma_3 > 0, \quad \text{or} \quad \gamma_1 > 0, \quad \text{or} \quad \gamma_2 < 0.
\]

The condition for the explosive solution and the critical point is

\[
\gamma_3 < 0, \quad \gamma_1 < 0, \quad \text{and} \quad \gamma_2 > 0,
\]

\[
s - t = \frac{1}{2\xi} \log \left( \frac{\gamma_2 + \xi}{\gamma_2 - \xi} \right).
\]

Remark A.1. By observation, if \(p < 0\), the conditions \((A.1)\) and \((A.2)\) hold, and we have \(a(t; s) \leq 0\) is a bounded solution as well as \(1 - 2a(t; s) \bar{\Omega}(t) > 1 > 0\) and \(1 - f(t; s) \bar{\Omega}(t) > 1 > 0\), hence we can
finally conclude the solutions of ODEs (4.24), (4.25), (4.26) are all bounded on $0 \leq t \leq s \leq T$. We also notice that $A(t) = \frac{a(t)}{(1-p)(1-2a(t)f(t))} \leq 0$, on $0 \leq t \leq s \leq T$.

A.2. The Hyperbolic Solution. The condition for the Hyperbolic solution is

$$
\Delta \triangleq \lambda^2 - \frac{2\lambda pp\sigma_\mu}{(1-p)\sigma_S^2} - \frac{p\sigma_\mu^2}{(1-p)\sigma_S^2} = 0,
$$

together with

$$
\gamma_2 = -\lambda + \frac{pp\sigma_\mu}{(1-p)\sigma_S} \neq 0,
$$

Then we can solve (4.27), (4.28), (4.29), (4.30) and (4.31) as:

$$
a(t; s) = \frac{-1}{2\gamma_1(s-t-\frac{1}{\gamma_2})} - \frac{\gamma_2}{2\gamma_1},
$$

$$
b(t; s) = -\frac{2\lambda\mu}{4\gamma_1\gamma_2(s-t-\frac{1}{\gamma_2})} - \frac{\gamma_2\lambda\mu(s-t+\frac{1}{\gamma_2})}{2\gamma_1},
$$

$$
c(t; s) = \frac{\gamma_2\sigma_\mu^2(s-t)}{2\gamma_1} + \frac{\lambda^2\sigma_\mu^2(s-t-\frac{1}{\gamma_2})(s-t)^3}{24\gamma_1(s-t-\frac{1}{\gamma_2})} + \frac{\sigma_\mu^2 \lambda \log |s-t\gamma_2 - 2|}{2\gamma_1},
$$

$$
f(t; s) = -\frac{1}{2\sigma_S} \left( \frac{1 - e^{2\xi_1(t-s)}}{\sigma_S \xi_1 + \lambda \sigma_S + \rho \sigma_\mu} + \frac{(\sigma_S \xi_1 - \lambda \sigma_S - \rho \sigma_\mu)e^{2\xi_1(t-s)}}{2\sigma_S \xi_1 e^{\xi_1(t-s)}} \right) - \frac{(\lambda \sigma_S + \rho \sigma_\mu)}{2\sigma_S} (s-t)
$$

$$
+ \frac{\sigma_\mu^2(1-\rho^2)}{2\gamma_1} \left[ \log |1 + \gamma_2(t-s)| - \gamma_2(s-t) \right].
$$

The condition for the bounded Hyperbolic solution is

$$
\gamma_2 < 0.
$$

The condition for the explosive solution and the critical point is

$$
\gamma_2 > 0, \quad \text{and} \quad s - t = \frac{1}{\gamma_2}.
$$

A.3. The Polynomial solution. The condition for the Polynomial solution is

$$
\Delta \triangleq \lambda^2 - \frac{2\lambda pp\sigma_\mu}{(1-p)\sigma_S^2} - \frac{p\sigma_\mu^2}{(1-p)\sigma_S^2} = 0,
$$

together with

$$
\gamma_2 = -\lambda + \frac{pp\sigma_\mu}{(1-p)\sigma_S} = 0,
$$

Then we can solve (4.27), (4.28), (4.29), (4.30) and (4.31) as:

$$
a(t; s) = \frac{p}{2(1-p)\sigma_S^2} (s-t),
$$

$$
b(t; s) = \frac{p}{2(1-p)\sigma_S^2} \lambda \mu (s-t)^2,
$$

$$
c(t; s) = -\frac{p}{4(1-p)\sigma_S^2} \sigma_\mu^2 (s-t)^2 + \frac{p}{6(1-p)\sigma_S^2} \lambda^2 \mu^2 (s-t)^3,
$$
The condition for the Tangent solution is

\[ \Delta \triangleq \lambda^2 - \frac{2\lambda \rho \sigma_\mu}{(1-p)\sigma_S^2} - \frac{p\sigma_\mu^2}{(1-p)\sigma_S^2} < 0, \]

Now, we define

\[ \zeta = \sqrt{-\Delta}, \quad \varpi = \tan^{-1} \left( \frac{\gamma_2}{\zeta} \right), \]

Then we can solve (4.27), (4.28), (4.29), (4.30) and (4.31) as:

\[
\begin{align*}
a(t; s) &= \frac{\zeta}{2\gamma_1} \tan \left( \zeta(s-t) + \varpi \right) - \frac{\gamma_2}{2\gamma_1}, \\
b(t; s) &= \frac{\lambda \mu}{\gamma_1} \left[ -1 - \tan(\varpi) \tan(\zeta(s-t) + \varpi) + \sec(\varpi) \sec(\zeta(s-t) + \varpi) \right], \\
c(t; s) &= \frac{2\lambda^2 \mu^2 \gamma_2 \sqrt{\gamma_2^2 + \zeta^2}}{2\gamma_1 \zeta} \left[ \sec(\varpi) - \sec(\zeta(s-t) + \varpi) \right] \\
&\quad + \frac{\lambda \mu^2 (2\gamma_2 + \zeta^2)}{2\gamma_1 \zeta^3} \left[ \tan(\zeta(s-t) + \varpi) - \tan(\varpi) \right] \\
&\quad - \frac{\lambda^2 \mu^2 (\gamma_2^2 + \zeta^2) - \gamma_2 \zeta^2 \sigma_\mu^2}{2\gamma_1 \zeta^2} + \frac{\sigma_\mu^2}{2\gamma_1} \log \left( \sec(\varpi) \cos(\zeta(s-t) + \varpi) \right), \\
f(t; s) &= -\frac{1}{2\sigma_S} \frac{1 - e^{2\xi_1(t-s)}}{(\sigma_S \xi_1 + \lambda \sigma_S + \rho \sigma_\mu) + (\sigma_S \xi_1 - \lambda \sigma_S - \rho \sigma_\mu)e^{2\xi_1(t-s)}}, \\
g(t; s) &= \frac{1}{2} \log \left( \frac{(\sigma_S \xi_1 + \lambda \sigma_S + \rho \sigma_\mu) + (\sigma_S \xi_1 - \lambda \sigma_S - \rho \sigma_\mu)e^{2\xi_1(t-s)}}{2\sigma_S \xi_1 e^{\xi_1(t-s)}} \right) - \frac{(\lambda \sigma_S + \rho \sigma_\mu)}{2\sigma_S} (s-t) \\
&\quad - \frac{\sigma_\mu^2 (1 - \rho^2)}{4} \left[ \frac{1}{2\gamma_1} \log \left| \frac{\cos(\zeta(t-s) + \varpi)}{\cos(\varpi)} \right| - \frac{\gamma_2}{2\gamma_1} (s-t) \right].
\end{align*}
\]

All Tangent solutions are explosive solutions and the critical point is

\[ s - t = \frac{\pi}{2\zeta} - \frac{1}{\zeta} \tan^{-1} \left( \frac{\gamma_2}{\zeta} \right). \]

References


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