1. For $\frac{1}{p} + \frac{1}{q} = 1$, let $S = \{ f \in L^p(\mathbb{R}) : \text{support}(f) \subset [-1, 1], \|f\|_{L^p} \leq 1 \}$, and let $g$ be a fixed but arbitrary function in $L^q(\mathbb{R})$, with support($g$) $\subset [-1, 1]$. Show that the image of $S$ under the map $f \mapsto f \ast g$ is a compact set in $C^0([-2, 2])$.

2. Let $f_1, f_2, f_3, \ldots$ be nonnegative Lebesgue-integrable functions on $\mathbb{R}^n$, such that

$$\sum_{k=1}^{\infty} \int (f_k - f_{k-1})^+ < \infty, \quad \lim_{k \to \infty} \int f_k = 0.$$ 

Show that $\lim \sup_{k \to \infty} f_k \equiv 0$ almost everywhere.

3. Let $1 < p < \infty$ and $f(x) = |x|^{-n/p}$ for $x \in \mathbb{R}^n$. Prove that $f$ is not the limit of a sequence $f_k \in C^\infty_0(\mathbb{R}^n)$ in the sense of convergence in $L^p_{\text{weak}}(\mathbb{R}^n)$. 

(That is, $\lim \sup_{k \to \infty} (\sup_{\lambda > 0} \lambda^p |\{x \in \mathbb{R}^n : |f(x) - f_k(x)| > \lambda\}|) > 0$ for any such sequence.)

4. Let $\mu$ be a Borel measure on $[0, 1]$. Assume that
   a) $\mu$ and Lebesgue measure are mutually singular.
   b) $\mu([0, t])$ depends continuously on $t$.
   c) For any function $f : [0, 1] \to \mathbb{R}$, if $f \in L^1(\text{Lebesgue})$ then $f \in L^1(\mu)$.
   (Note that $f$ has a finite value at every point.)

Show that $\mu \equiv 0$. 