1. Let $X$, $Y$, and $Z$ be NLS’s and $T : X \times Y \to Z$ a bilinear map.
   (a) Prove that the following are equivalent.

   (i) $T$ is continuous;
   (ii) $T$ is continuous at $(0, 0)$;
   (iii) $T$ is bounded, meaning that there is some $M \geq 0$ such that
   $$\|T(x, y)\|_Z \leq M \|x\|_X \|y\|_Y \quad \forall x \in X, \ y \in Y.$$  

   (b) Show that the minimal $M$ above gives a norm on the set of continuous bilinear maps. 
   That is, $\|\cdot\|$ is a norm, where
   $$\|T\| = \sup_{x \in X, y \in Y} \frac{\|T(x, y)\|_Z}{\|x\|_X \|y\|_Y}.$$  

2. Let $H$ be a nontrivial Hilbert space. Let $P : H \to M$ be a linear projection operator, and let $Q : H \to N$ be an orthogonal projection operator. Assume that $M$ and $N$ are neither $\{0\}$ nor $H$.
   (a) Prove that $\|P\| \geq 1$.
   (b) Prove that $\|P\| = 1$ if and only if $P$ is an orthogonal projection.
   (c) Suppose now that $P$ is an orthogonal projection, and also that $PQ = QP$. Show that $PQ$ is an orthogonal projection onto $M \cap N$.

3. Let $K : L^2(0, 1) \to L^2(0, 1)$ be the integral operator defined as
   $$Ku(x) = \int_0^1 e^{x-y}u(y)dy.$$  

   (a) Find the range of $K$. Is the range of $K$ closed? Is $K$ a compact operator?
   (b) Compute the adjoint operator $K^*$, and find its kernel.
   (c) Verify explicitly that $Ku = f$ is solvable if and only if $f \perp \text{Ker} \ K^*$.  

Applied Math Prelim
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Part I
Choose any 3 of the 4 following problems.

4. For \( \varphi \in C^0(\mathbb{R}^d) \), let the restriction map \( R : C^0(\mathbb{R}^d) \to C^0(\mathbb{R}^{d-k}) \) be defined by 
\[ R\varphi(x') = \varphi(x',0), \quad \forall x' \in \mathbb{R}^{d-k} \text{ and } 0 \in \mathbb{R}^k, \] for \( 0 < k < d \), with \( k \) an integer number.

Show that the restriction map \( R \) extends to a bounded linear map from \( H^s(\mathbb{R}^d) \) onto \( H^{s-k/2}(\mathbb{R}^{d-k}) \), provided that \( s > k/2 \).

Hint: Show this result first for the restriction of functions in \( S(\mathbb{R}^d) \) using the Sobolev norms involving the Fourier representation.

5. Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain with a Lipschitz boundary, \( f \in L^2(\Omega) \) and \( \alpha > 0 \). Consider the Robin boundary value problem in \( \Omega \),
\[
\begin{cases}
-\Delta u + u = f & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
(a) For this problem, formulate a variational principle \( B(u,v) = (f,v), \forall v \in H^1(\Omega) \).
(b) Show that this problem has a unique weak solution.

6. Set up and apply the contraction mapping principle to show that the boundary value problem \( (\varepsilon > 0) \):
\[
\begin{cases}
-\varepsilon u + u - \varepsilon u^2 = f(x), \quad x \in (0, +\infty), \\
u(0) = 1, \quad \lim_{x \to +\infty} u(x) = 0,
\end{cases}
\]
where \( f(x) \) is a smooth compactly supported function on \( (0, +\infty) \), has a unique smooth solution if \( \varepsilon \) is small enough.

7. Show that for \( y \in \mathbb{R}^2 \) fixed, \( \frac{1}{2\pi} \ln |x - y| \) is locally integrable in \( \mathbb{R}^2 \), i.e. it is a function in \( L_{1,\text{loc}}(\mathbb{R}^2) \); and that it is a fundamental solution of \( \Delta u = \delta_y \), where \( \Delta = \partial^2_{x_1} + \partial^2_{x_2} \) is the Laplace operator.