1. Let \( f(x) \) be a smooth function.

(a) Write down the Newton’s method for root finding.

(b) Assume that \( f(\xi) = f'(\xi) = f''(\xi) = 0 \). If \( (x_k) \) is a sequence obtained by Newton’s method, prove that

\[
\xi - x_{k+1} = (\xi - x_k) \left( 1 - \frac{1}{3} \frac{f'''(\alpha_k)}{f''(\beta_k)} \right)
\]

where \( \alpha_k \) and \( \beta_k \) lie between \( \xi \) and \( x_k \).

(c) Suppose \( 0 < m < |f'''(x)| < M \) in a neighborhood \( [\xi - \eta, \xi + \eta] \) for \( \eta > 0 \). Find a condition on \( m \) and \( M \) under which the Newton method converges for any initial condition from \( [\xi - \eta, \xi + \eta] \). State the convergence rate.

2. Suppose that \( A \) is a matrix of size \( m \times n \) with \( m > n \).

(a) Show how to solve the least square problem

\[
\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - b\|^2
\]

using the QR decomposition and the singular value decomposition (SVD), respectively. Notice that the rank of \( A \) may be smaller than \( n \).

(b) Show how to solve the regularized problem

\[
\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - b\|^2 + \frac{1}{2} \alpha \|x\|^2
\]

with \( \alpha > 0 \) using the singular value decomposition (SVD).

3. Suppose that \( P_1(x), \ldots, P_n(x) \) are \( n \) linear independent functions on a domain \( \Omega \).

(a) Suppose that \( (x_j, u_j) \) for \( j = 1, 2, \ldots, n \) are given for \( \{x_j\} \) different. What is the linear system for finding \( u(x) \in \text{span}\{P_1(x), \ldots, P_n(x)\} \) that interpolates \( (x_j, u_j) \), i.e., \( u(x_j) = u_j \).

(b) Suppose that \( g(x) \) is a function defined on \( \Omega \). What is the linear system for finding \( u(x) \in \text{span}\{P_1(x), \ldots, P_n(x)\} \) that solves

\[
\min_u \frac{1}{2} \int_{\Omega} |g(x) - u(x)|^2 \, dx.
\]

(c) Assume that \( \Omega \) is the periodic interval \( [0, 1] \), \( x_j = (j - 1)/n \) for \( j = 1, \ldots, n \), and \( \{P_i(x), i = 1, \ldots, n\} = \{e^{2\pi\sqrt{-1}kx}, k = -m, \ldots, m\} \) where \( n = 2m + 1 \). Write down explicitly the linear system of the previous two questions.

(d) How to solve these two problems efficiently in \( O(n \log n) \) steps? [Hint: Use trapezoidal rule (for quadrature) and FFT.]